Weak extinction versus global exponential growth of total mass for superdiffusions

János Engländer, Yan-Xia Ren* and Renming Song†

Abstract

Consider a superdiffusion X on $\mathbb{R}^d$ corresponding to the semilinear operator $\mathcal{A}(u) = Lu + \beta u - ku^2$, where L is a second order elliptic operator, $\beta(\cdot)$ is in the Kato class and bounded from above, and $k(\cdot) \geq 0$ is bounded on compact subsets of $\mathbb{R}^d$ and is positive on a set of positive Lebesgue measure.

The main purpose of this paper is to complement the results obtained in [8], in the following sense. Let λ_∞ be the L^∞-growth bound of the semigroup corresponding to the Schrödinger operator $L+\beta$. If $\lambda_\infty \neq 0$, then we prove that, in some sense, the exponential growth/decay rate of $\|X_t\|$, the total mass of X_t, is λ_∞. We also describe the limiting behavior of $\exp(-\lambda_\infty t)\|X_t\|$ in these cases. This should be compared to the result in [8], which says that the generalized principal eigenvalue λ_2 of the operator gives the rate of *local* growth when it is positive, and implies local extinction otherwise. It is easy to show that $\lambda_\infty \geq \lambda_2$, and we discuss cases when $\lambda_\infty > \lambda_2$ and when $\lambda_\infty = \lambda_2$.

When $\lambda_\infty = 0$, and under some conditions on β, we give a sufficient and necessary condition for the superdiffusion X to exhibit weak extinction. We show that the branching intensity k affects weak extinction; this should be compared to the known result that k does not affect weak *local* extinction (which only depends on the sign of λ_2, and which turns out to be equivalent to local extinction) of X.

Contents

*The research of this author is supported by NSFC (Grant No. 10971003, 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education.

†Research supported in part by a grant from the Simons Foundation (208236).

AMS Subject Classifications (2000): Primary 60J80; Secondary 60G57, 60J45

Keywords and Phrases: superdiffusion, superprocess, measure-valued process, gauge theorem, Kato class, growth bound, principal eigenvalue, h-transform, weak extinction, total mass.

1 Introduction

1.1 Model

For any positive integer i and $\eta \in (0,1]$, let $C^{i,\eta}(\mathbb{R}^d)$ denote the space of i times continuously differentiable functions with all their i-th order derivatives belonging to $C^\eta(\mathbb{R}^d)$. (Here $C^\eta(\mathbb{R}^d)$ denotes the usual Hölder space.) For any $x \in \mathbb{R}^d$, we will use $\{\xi_t, \Pi_x, t \geq 0\}$ to denote the L-diffusion with $\Pi_x(\xi_0 = x) = 1$, where

$$L := \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \quad \text{ on } \mathbb{R}^d,$$

and a, b satisfy the following

(1) the symmetric matrix $a = \{a_{i,j}\}$ satisfies

$$A_1|v|^2 \le \sum_{i,j=1}^{d} a_{i,j}(x)v_iv_j \le A_2|v|^2, \quad \text{for all } v \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d$$

with some $A_1, A_2 > 0$, and $a_{i,j} \in C^{1,\eta}, i, j = 1, \cdots, d$, for some η in $(0, 1]$;

(2) the coefficients b_i, $i = 1, \cdots, d$, are measurable functions satisfying

$$\sum_{i=1}^{d} |b_i(x)| \le C(1 + |x|), \qquad \text{for all } x \in \mathbb{R}^d$$

with some $C > 0$;

(3) there exists a differentiable function $Q : \mathbb{R}^d \to \mathbb{R}$ such that $b = a\nabla Q$.

Remark 1.1 Under (1)–(3) above, the diffusion process ξ is *conservative* on $\mathbb{R}^d$. That is,

$$\Pi_x(\xi_t \in \mathbb{R}^d,\ \forall t > 0) = 1,$$

for all $x \in \mathbb{R}^d$; equivalently, the semigroup corresponding to ξ leaves the function $f \equiv 1$ invariant. For a proof, see, for instance, [28, Theorem 10.2.2].

Define

$$m(x) = e^{2Q(x)}, \qquad x \in \mathbb{R}^d. \tag{1.1}$$

Then ξ is an m-symmetric Markov process, that is, the semigroup of ξ in $L^2(\mathbb{R}^d, m(x)\mathrm{d}x)$ is symmetric. If $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions with compact support, then the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ of ξ in $L^2(\mathbb{R}^d, m(x)\mathrm{d}x)$ is the closure of the form given by

$$\mathcal{E}(u, v) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla u a \nabla v)\exp(2Q)\mathrm{d}x, \quad u, v \in C_c^\infty(\mathbb{R}^d).$$

For any measurable space $(E, \mathcal{B})$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}$, equipped with the weak topology. We denote by $\mathcal{M}$ the Borel σ-field on $M(E)$, and so $\mathcal{M}$ generated by all the functions $f_B(\mu) = \mu(B)$ with $B \in \mathcal{B}$. The space of finite measures with compact support will be denoted by $M_c(E)$. The expression $\langle f, \mu\rangle$ stands for the integral of f with respect to μ.

With β belonging to a certain Kato class (see Definition 1.2) and k being locally bounded from above and nonnegative, we will define the fundamental quantity λ_2 in

(1.4) and show that $\lambda_2 < \infty$. We will write $(\{X_t\}_{t\geq 0}; \mathbb{P}_\mu,\ \mu \in M(\mathbb{R}^d))$ to denote the *superprocess* (a measure-valued Markov process) with $\mathbb{P}_\mu(X_0 = \mu) = 1$, corresponding to the semilinear elliptic operator $\mathcal{A}(u) := Lu + \beta u - ku^2$ on $\mathbb{R}^d$. For the precise definition, see Definition 1.3 below. As we will see in Theorem 1.2 and Theorem 1.4, the superprocess is well defined and has a *càdlàg* version.

1.2 Motivation

The main purpose of this paper is to complement the results obtained in [8]. In particular, we study the growth/decay rate of the total mass of X and weak extinction of X. Whereas in [8], the local behavior of the mass has been shown to be intimately related to the generalized principal eigenvalue corresponding to the expectation operator, here we will show that the global behavior of the mass is linked to another important quantity, the L^∞-bound for the semigroup.

1.3 Known results

We first recall some definitions from Engländer and Kyprianou [8].

Definition 1.1 Fix $\mathbf{0} \neq \mu \in M(\mathbb{R}^d)$ with compact support.

(i) We say that X *exhibits local extinction* under $\mathbb{P}_\mu$ if for every bounded Borel set $B \subset \mathbb{R}^d$, there exists a random time τ_B such that

$$\mathbb{P}_\mu(\tau_B < \infty) = 1 \quad \text{and} \quad \mathbb{P}_\mu(X_t(B) = 0 \text{ for all } t \geq \tau_B) = 1.$$

(ii) We say that X *exhibits weak local extinction* under $\mathbb{P}_\mu$ if for every bounded Borel set $B \subset \mathbb{R}^d$, $\mathbb{P}_\mu(\lim_{t\to\infty} X_t(B) = 0) = 1$.

(iii) We say that X *exhibits extinction* under $\mathbb{P}_\mu$ if there exists a stopping time τ such that

$$\mathbb{P}_\mu(\tau < \infty) = 1 \quad \text{and} \quad \mathbb{P}_\mu(X_t(\mathbb{R}^d) = 0 \text{ for all } t \geq \tau) = 1.$$

(iv) We say that X *exhibits weak extinction* under $\mathbb{P}_\mu$ if $\mathbb{P}_\mu(\lim_{t\to\infty} X_t(\mathbb{R}^d) = 0) = 1$.

Let λ_2 be the growth bound of the semigroup in $L^2(\mathbb{R}^d, m)$ corresponding to the operator $L+\beta$ (see (1.4) and (1.5)). In [23], Pinsky gave a criterion for the local extinction of X under the assumption that β is Hölder continuous, namely, he proved that X exhibits local extinction if and only if $\lambda_2 \leq 0$. In particular, local extinction does not depend on the branching intensity k, but it does depend on L and β. (Note that, in regions where $\beta > 0$, β can be considered as mass creation, whereas in regions where $\beta < 0$, β can be considered as mass annihilation.) Since local extinction depends on the sign of

λ_2, therefore, heuristically, it depends on the competition between the outward speed of particles and the mass creation. The main tools of [23] are PDE techniques.

In [8], Engländer and Kyprianou presented probabilistic (martingale and spine) arguments for the fact that $\lambda_2 \leq 0$ implies weak local extinction, while $\lambda_2 > 0$ implies that, for any $\lambda < \lambda_2$ and any nonempty relatively compact open set B,

$$\mathbb{P}_\mu \left(\limsup_{t\to\infty} e^{-\lambda t} X_t(B) = \infty \right) > 0$$

holds for any nonzero initial measure μ.

Putting things together, one concludes that in this case *local extinction is in fact equivalent to weak local extinction* and there is a dichotomy in the sense that the process either exhibits local extinction (when $\lambda_2 \leq 0$), or there is local exponential growth with positive probability (when $\lambda_2 > 0$).

We will see that, on the other hand, extinction and weak extinction are different in general. The intuition behind this is that the total mass $\|X_t\|$ may stay positive but decay to zero, *while drifting out* (local extinction) and on its way obeying changing branching laws. (For a concrete example see Example 5.3.) This could not be achieved in a fixed compact region with fixed branching coefficients.

In [8] an analogous result has been verified for branching diffusions too, by using the same method. (Note that for branching diffusions, weak (local) extinction and (local) extinction are obviously the same, because the local/total mass is an integer.) It was also noted that the growth rate of the total mass may exceed λ_2 (see [8, remark 4]).

1.4 Our main results

It is important to point out that weak extinction, unlike local extinction, depends on the branching intensity k as well. It is natural to ask whether β or k plays the more important role. The answer: β plays the main role, k only has a minor significance. We will prove that the exponential growth rate of the total mass is λ_∞, defined by (1.8). More precisely, there are three cases:

1. If mass creation is large enough so that $\lambda_\infty > 0$, then the total mass of X tends to infinity exponentially with rate $\lambda_\infty > 0$;

2. if annihilation is strong enough so that $\lambda_\infty < 0$, then the total mass of X tends to zero exponentially with rate $\lambda_\infty < 0$, even under survival;

3. if $\lambda_\infty = 0$, then weak extinction depends on k.

Concerning the third case, under some further conditions on β, we will give a necessary and sufficient condition for X to exhibit weak extinction (see Remark 1.10).

In all the work mentioned above, β is assumed to be Hölder continuous. In this paper, we relax this condition by using results of [2, 4, 13, 14, 30] on Schrödinger operators. The results of this paper are new even under the assumption that β is Hölder continuous. Furthermore, even under the Hölder continuity assumption, the arguments of this paper can not be simplified by much.

Before we give the main results of this paper, let us introduce some definitions and notations.

Definition 1.2 (Kato class) A measurable function q on $\mathbb{R}^d$ is said to be in the *Kato class* $\mathbf{K}(\xi)$ if

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\Pi_x\left(\int_0^t |q(\xi_s)|\mathrm{d}s\right)=0.$$

It is easy to see that any bounded function is in the Kato class $\mathbf{K}(\xi)$. For any $q\in\mathbf{K}(\xi)$, denote

$$e_q(t):=\exp\left(\int_0^t q(\xi_u)\mathrm{d}u\right), \tag{1.2}$$

and define

$$e_q(\infty):=\exp\left(\int_0^\infty q(\xi_s)\mathrm{d}s\right), \tag{1.3}$$

whenever the integral on the righthand side makes sense.

Assumption 1.1 In the remainder of this article, we will always assume that $\beta\in\mathbf{K}(\xi)$.

One may define a semigroup $\{P_t^\beta\}_{t\ge 0}$ on L^p, for any $p\in[1,\infty]$, by

$$P_t^\beta f(x):=\Pi_x[e_\beta(t)f(\xi_t)].$$

For any $p\in[1,\infty]$, $\|\cdot\|_{p,p}$ stands for the operator norm from $L^p(\mathbb{R}^d,m)$ to $L^p(\mathbb{R}^d,m)$. It follows from [5, Theorem 3.10] that, for any $t>0$ and $p\in[1,\infty)$, $\|P_t^\beta\|_{p,p}\le\|P_t^\beta\|_{\infty,\infty}\le e^{c_1t+c_2}$ for some constants c_1,c_2, and that $\{P_t^\beta\}_{t\ge0}$ is a strongly continuous semigroup in $L^p(\mathbb{R}^d,m)$ for any $1\le p<\infty$. We define

$$\lambda_2(\beta):=\lim_{t\to\infty}\frac{1}{t}\log\|P_t^\beta\|_{2,2}. \tag{1.4}$$

In Section 2 we will prove the following probabilistic characterization of $\lambda_2(\beta)$

$$\lambda_2(\beta)=\sup_{A\subset\subset\mathbb{R}^d}\lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in A}\Pi_x\left(e_\beta(t);\tau_A>t\right). \tag{1.5}$$

(Here $A \subset\subset \mathbb{R}^d$ means that A is a bounded set in $\mathbb{R}^d$.) In general, when β is Hölder-continuous, $\lambda_2(\beta)$ coincides with the so-called *generalized principal eigenvalue* of $L+\beta$ defined in [22] . In our symmetric setting however, for such a β, the situation is even simpler: $\lambda_2(\beta)$ is the *supremum of the L^2-spectrum* for the self-adjoint realization of the symmetric operator $L+\beta$ on $\mathbb{R}^d$, obtained via the Friedrichs extension theorem. (See Chapter 4, and especially Proposition 10.1 in [22] for more explanation).

Definition 1.3 (The (L,β,k)-superprocess) An (L,β,k)-*superprocess* is a measure-valued Markov process $(\{X_t\}_{t\geq 0}; \mathbb{P}_\mu,\ \mu \in M(\mathbb{R}^d))$ such that $\mathbb{P}_\mu(X_0=\mu)=1$, and for any bounded Borel $f \geq 0$ on $\mathbb{R}^d$, one has

$$\mathbb{P}_\mu \exp\langle -f, X_t\rangle = \exp\langle -u(t,\cdot), \mu\rangle, \tag{1.6}$$

where u is the minimal nonnegative solution to

$$u(t,x) + \Pi_x \int_0^t k(\xi_s)(u(t-s,\xi_s))^2 \mathrm{d}s - \Pi_x \int_0^t \beta(\xi_s)u(t-s,\xi_s)\mathrm{d}s = \Pi_x f(\xi_t). \tag{1.7}$$

We will also say that $(\{X_t\}_{t\geq 0}; \mathbb{P}_\mu,\ \mu \in M(\mathbb{R}^d))$ is the superprocess 'corresponding to the semilinear elliptic operator $\mathcal{A}(u) := Lu + \beta u - ku^2$ on $\mathbb{R}^d$.'

Theorem 1.2 *Suppose that $\beta \in \mathbf{K}(\xi)$, and $k \geq 0$ is locally bounded. The (L,β,k)-superprocess exists.*

Remark 1.3 (Minimality and uniqueness) Under our general condition on k, we do not claim the uniqueness of the solution to the cumulant equation (1.7). In the Appendix, we will construct a minimal solution instead. If, however, $k \in \mathbf{K}(\xi)$ holds as well, then the solution is unique, see Remark 6.1.

Right after the construction of the superprocess, one of course would like to know what regularity properties of the paths one can assume.

Theorem 1.4 *Suppose that $\beta \in \mathbf{K}(\xi)$ and is bounded from above, and $k \geq 0$ is locally bounded. The superprocess constructed in Theorem 1.2 has a version which has càdlàg paths (that is, right continuous paths with left limits in the weak topology of measures).*

Throughout this paper, the following assumption will be in force:

Assumption 1.2 (Regularity assumption) The superprocess X has *càdlàg* paths.

The proofs of Theorems 1.2 and 1.4 are relegated to the Appendix.

Remark 1.5 (β unbounded from above) Note that to get a regular version of X we supposed that β is bounded from above. This was a purely technical assumption, and in fact, *this is the only reason we need this condition.* All of the arguments in this paper work for any $\beta \in \mathbf{K}(\xi)$ except when we need the regularity of X.

Returning now to the analytic tools needed, another very important quantity besides λ_2 is given in the following definition.

Definition 1.4 (L^∞-growth bound) Define

$$\lambda_\infty(\beta) := \lim_{t\to\infty} \frac{1}{t} \log \|P_t^\beta\|_{\infty,\infty} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t). \tag{1.8}$$

We call $\lambda_\infty = \lambda_\infty(\beta)$ the L^∞*-growth bound.*

As already mentioned, λ_∞ plays a crucial role in describing the behavior of the *total* mass of the superprocess, while λ_2 describes the behavior of the *local* mass. Indeed from (1.4) and (1.8) it is obvious that $\lambda_\infty(\beta) \geq \lambda_2(\beta)$.

In fact, $\lambda_\infty(\beta) = \lambda_2(\beta)$ and $\lambda_\infty(\beta) > \lambda_2(\beta)$ are both possible. For conditions under which $\lambda_\infty(\beta) = \lambda_2(\beta)$, we refer to Chen [3, Section 4] and the references therein. We will give some examples of $\lambda_\infty(\beta) > \lambda_2(\beta)$ in Section 5.

For simplicity, we will write $\lambda_2(\beta)$ as λ_2, and $\lambda_\infty(\beta)$ as λ_∞ when the potential β is fixed.

The following notion is of fundamental importance.

Definition 1.5 (gauge function) For any $\beta \in \mathbf{K}(\xi)$, we define

$$g_\beta(x) = \Pi_x(e_\beta(\infty)), \quad x \in \mathbb{R}^d, \tag{1.9}$$

when the right hand side is well defined. The function g_β, called the *gauge function*, is very useful in studying the potential theory of the Schrödinger-type operator $L + \beta$.

We are now ready to state the main results of this paper, the first of which treats the 'overscaling' and 'underscaling' of the total mass $\|X_t\| := \langle 1, X_t\rangle$.

Theorem 1.6 *Suppose $\mu \in M(\mathbb{R}^d)$ and $\mu \neq \mathbf{0}$.*

(1) For any $\lambda > \lambda_\infty$,

$$\mathbb{P}_\mu\left(\lim_{t\to\infty} e^{-\lambda t}\|X_t\| = 0\right) = 1. \tag{1.10}$$

In particular, if $\lambda_\infty < 0$, then X suffers weak extinction.

(2) Assume that k is bounded. If $\lambda_\infty > 0$ and

$$\liminf_{t\to\infty} \Pi_x e_\beta(t) / \sup_{y\in\mathbb{R}^d} \Pi_y e_\beta(t) > 0 \quad \text{for all } x \in \mathbb{R}^d \tag{1.11}$$

holds, then for any $\lambda < \lambda_\infty$,

$$\mathbb{P}_\mu\left(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| = \infty\right) > 0. \tag{1.12}$$

The next two theorems give some insight as to what happens when the scaling of the total mass is exactly at λ_∞.

Theorem 1.7 (Scaling at λ_∞) *Suppose $\mu \in M(\mathbb{R}^d)$ and $\mu \neq \mathbf{0}$.*

(1) Assume that $\lambda_\infty > 0$ and that (1.11) holds.

If

$$\lim_{t\to\infty} \Pi_x e_{\beta-\lambda_\infty}(t) = \infty \quad \text{for all } x \in \mathbb{R}^d, \tag{1.13}$$

then

$$\mathbb{P}_\mu\left(\limsup_{t\to\infty} e^{-\lambda_\infty t}\|X_t\| = \infty\right) > 0. \tag{1.14}$$

(2) If $g_{\beta-\lambda_\infty}(x) \equiv 0$ in $\mathbb{R}^d$ and

$$\sup_{x\in\mathbb{R}^d} \Pi_x\left(\sup_{t\geq 0} e_{\beta-\lambda_\infty}(t)\right) < \infty, \tag{1.15}$$

then

$$\mathbb{P}_\mu\left(\liminf_{t\to\infty} e^{-\lambda_\infty t}\|X_t\| = 0\right) = 1. \tag{1.16}$$

If, in addition, $\beta \leq 0$ on $\mathbb{R}^d$, the superprocess suffers weak extinction.

Theorem 1.8 *Assume that there is a bounded solution $h > 0$ of $(L + \beta - \lambda_\infty)h = 0$ in $\mathbb{R}^d$ in the sense of distributions. If there exists an $x_0 \in \mathbb{R}^d$ such that*

$$\Pi_{x_0}\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)\mathrm{d}s < \infty, \tag{1.17}$$

then $\lim_{t\to\infty} e^{-\lambda_\infty t}\langle h, X_t\rangle$ exists $\mathbb{P}_\mu$-a.s. and in $L^1(\mathbb{P}_\mu)$, and $\mathbb{P}_\mu(\|X_t\| > 0,\ \forall t > 0) > 0$ for all nontrivial measure $\mu \in M_c(\mathbb{R}^d)$.

If, in addition, h satisfies that

$$\inf_{x\in\mathbb{R}^d} h(x) > 0, \tag{1.18}$$

then the scaling at λ_∞ is the correct one in the sense that for every nontrivial $\mu \in M_c(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\left(\limsup_{t\to\infty} e^{-\lambda_\infty t}\|X_t\| < \infty\right) = 1 \tag{1.19}$$

and

$$\mathbb{P}_\mu\left(\liminf_{t\to\infty} e^{-\lambda_\infty t}\|X_t\| > 0\right) > 0. \tag{1.20}$$

Now we give a partial converse of Theorem 1.8. To state this result, we need to introduce another function class.

Definition 1.6 (The class $\mathbf{K}_\infty(\xi)$) Suppose that ξ is transient. A function $q \in \mathbf{K}(\xi)$ is *said to be in the class* $\mathbf{K}_\infty(\xi)$ if for any $\epsilon > 0$ there exist a compact set K and a constant $\delta > 0$ such that for any subset A of K with $m(A) < \delta$,

$$\sup_{x\in\mathbb{R}^d}\int_{(\mathbb{R}^d\setminus K)\cup A} \widetilde{G}(x,y)|q(y)|m(y)\mathrm{d}y < \epsilon, \tag{1.21}$$

where m is the function defined in (1.1) and $\widetilde{G}(x,y)$ is the Green function corresponding to ξ with respect to $m(x)\mathrm{d}x$ in $\mathbb{R}^d$.

The class $\mathbf{K}_\infty(\xi)$ was first introduced in [4, 2]. When ξ is transient and $\beta \in \mathbf{K}_\infty(\xi)$, we have $\lambda_\infty \geq 0$. In fact, it follows from [4, Proposition 2.1] that $\Pi_x\left(\int_0^\infty |\beta|(\xi_s)ds\right)$ is bounded in $\mathbb{R}^d$. Let M be the upper bound. By Jensen's inequality, we have

$$\Pi_x e_\beta(t) \geq \exp\left(-\Pi_x\int_0^\infty |\beta|(\xi_s)\mathrm{d}s\right) \geq e^{-M}, \tag{1.22}$$

which implies that

$$\frac{1}{t}\log\sup_{x\in\mathbb{R}^d}\Pi_x e_\beta(t) \geq -M/t.$$

Thus by definition,

$$\lambda_\infty = \lim_{t\to\infty}\sup_{x\in\mathbb{R}^d}\frac{1}{t}\log\Pi_x e_\beta(t) \geq 0.$$

Note that (1.22) implies that $g_\beta \geq e^{-M}$. It follows from the gauge theorem (see [4, Theorem 2.2] or [2, Theorem 2.6]) that, if ξ is transient and $\beta \in \mathbf{K}_\infty(\xi)$, then g_β is either bounded or identically infinite. It follows from [2, Corollary 2.9] that the boundedness of g_β implies that $\sup_{x\in\mathbb{R}^d}\Pi_x(\sup_{t\geq 0} e_\beta(t)) < \infty$ for every $x \in \mathbb{R}^d$, and hence $\lambda_\infty(\beta) = 0$.

Theorem 1.9 (Weak extinction in the radial case) *Assume that k and β are radial functions, and L is radial (i.e., $a_{i,j}$, $i, j = 1, 2, \cdots, d$, and Q are radial functions). Suppose*

that ξ is transient, $\beta \in \mathbf{K}_\infty(\xi)$, and that $g_\beta(x)$ is not identically infinite (which implies that g_β is bounded and hence $\lambda_\infty = 0$). If

$$\Pi_x \left[\int_0^\infty e_\beta(s) k(\xi_s) \mathrm{d}s \right] = \infty \quad \text{for all } x \in \mathbb{R}^d, \tag{1.23}$$

then for every $\mu \in M(\mathbb{R}^d)$,

$$\mathbb{P}_\mu(\lim_{t\to\infty} \|X_t\| = 0) = 1. \tag{1.24}$$

Remark 1.10 *In particular, if ξ is transient, $\beta \in \mathbf{K}_\infty(\xi)$ and g_β is not identically infinite, then g_β is a solution of $(L+\beta)u = 0$ in the distribution sense, and is bounded between two positive numbers. In this case, Theorem 1.8 and Theorem 1.9 imply that condition* (1.23) *is a necessary and sufficient for X to exhibit weak extinction.*

In Section 5 we will give some examples for which the conditions of our theorems are satisfied.

2 Preparations: Feynman-Kac semigroups

Recall that β is in the Kato class $\mathbf{K}(\xi)$. In this section, we present some preliminary results on the Feynman-Kac semigroup. Recall from Section 1 that

$$P_t^\beta f(x) := \Pi_x[e_\beta(t) f(\xi_t)],$$

and that $\{P_t^\beta, t \geq 0\}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^d, m)$ for every $1 \leq p < \infty$.

For any domain $D \subset \mathbb{R}^d$ and $x \in D$, we will use $\delta_D(x)$ to denote the distance from x to D^c. Let ξ^D be the subprocess of ξ killed upon exiting D. We will use $\{P_t^{\beta,D}, t \geq 0\}$ to denote the semigroup of ξ^D:

$$P_t^{\beta,D} f(x) := \Pi_x[e_\beta(t) f(\xi_t), t < \tau_D],$$

where

$$\tau_D = \inf\{t > 0 : \xi_t \notin D\}.$$

When D^c is non-polar, that is, when $\Pi_x(\tau_D < \infty)$ is not identically zero, ξ^D is transient. In this case, we will write G_D to denote the Green function of ξ^D with respect to the Lebesgue measure. Then $\widetilde{G}_D(x,y) := G_D(x,y)/m(y)$ is the Green function of ξ^D with respect to $m(y)\mathrm{d}y$.

For any $n \geq 1$, put $D_n = B(0,n)$. We will use the shorthand $\xi^{(n)}$ to denote ξ^{D_n} and G_n to denote G_{D_n}. It follows from [15, 17] that G_n is comparable to the Green function

of the killed Brownian motion in D_n. Therefore, there exists $c_1 = c_1(n, d) > 1$ such that when $d \geq 3$,

$$c_1^{-1}\left(1 \wedge \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq G_B(x,y) \leq c_1 \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right), \quad x, y \in B \tag{2.1}$$

for any ball $B \subset D_n$; when $d = 2$

$$c_1^{-1} \log\left(1 + \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq G_B(x,y) \leq c_1 \log\left(1 + \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right), \quad x, y \in B \tag{2.2}$$

for any ball $B \subset D_n$; and when $d = 1$

$$c_1^{-1}(\delta_B(x) \wedge \delta_B(y)) \leq G_B(x,y) \leq c_1(\delta_B(x) \wedge \delta_B(y)), \quad x, y \in B \tag{2.3}$$

for any ball $B \subset D_n$.

2.1 The 3G inequalities and the Martin kernel

For convenience, we define

$$u(x) := \begin{cases} |x|^{2-d}, & d \geq 3 \\ \log|x|^{-1}, & d = 2 \\ |x|, & d = 1. \end{cases} \tag{2.4}$$

Using (2.1)–(2.3), we get that there exists $c = c(d, n)$ such that, when $d \geq 3$ (by [5, Theorem 6.5])

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \leq c(u(x-y) + u(y-z)), \quad x, y, z \in B \tag{2.5}$$

for any ball $B \subset D_n$; when $d = 2$ (by [5, Theorem 6.15])

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \leq c[(1 \vee u(x-y)) + (1 \vee u(y-z))], \quad x, y, z \in B \tag{2.6}$$

for any ball $B \subset D_n$; and when $d = 1$, by direct calculation,

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \leq c, \quad x, y, z \in B \tag{2.7}$$

for any ball $B \subset D_n$. The three inequalities above are called *3G inequalities*. For any ball B, let $M_B(x,z), (x,z) \in B \times \partial B$, be the Martin kernel of ξ^B:

$$M_B(x,z) := \lim_{B \ni y \to z \in \partial B} \frac{G_B(x,y)}{G_B(x_0,y)}$$

for some $x_0 \in B$. Then one can easily deduce from the 3G inequalities above that there exists $c = c(d, n) > 0$ such that, when $d \geq 3$,

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \leq c(u(x-y) + u(y-z)), \quad x, y \in B, z \in \partial B \tag{2.8}$$

for every ball $B \subset D_n$; when $d = 2$,

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \leq c[(1 \vee u(x-y)) + (1 \vee u(y-z))], \quad x, y \in B, z \in \partial B \tag{2.9}$$

for every ball $B \subset D_n$; when $d = 1$,

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \leq c, \quad x, y \in B, z \in \partial B \tag{2.10}$$

for every ball $B \subset D_n$.

It follows from [16, 17] that for any $n \geq 1$, there exist $c_i = c_i(n) > 1$, $i = 1, 2$, such that the heat kernel $p_t^{(n)}$ associated with $\xi^{(n)}$ satisfies

$$\begin{aligned} & c_1^{-1} t^{-d/2} \left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right) e^{-\frac{c_2|x-y|^2}{t}} \leq p_t^{(n)}(x,y) \\ & \leq c_1 t^{-d/2} \left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right) e^{-\frac{|x-y|^2}{c_2 t}} \end{aligned} \tag{2.11}$$

for all $(t, x, y) \in (0, 1] \times D_n \times D_n$.

We then have the following result.

Proposition 2.1 *If $\beta \in \mathbf{K}(\xi)$, then for any $n \geq 1$,*

$$\lim_{r \to 0} \sup_{x \in D_n} \int_{|y-x|<r} u(x-y)|\beta(y)| \mathrm{d}y = 0.$$

Proof. It follows from (2.11) that there exist constants $c_1, c_2 > 1$ such that for any $(t, x, y) \in (0, 1] \times D_n \times D_n$,

$$p_t^{(n+1)}(x,y) \geq c_1^{-1} \exp\left\{-\frac{c_2|x-y|^2}{t}\right\}.$$

Since

$$\int_0^t \Pi_x[|\beta(\xi_s)|] \mathrm{d}s \geq \int_0^t \int_{D_n} p_s^{(n+1)}(x,y)|\beta(y)| \mathrm{d}y \mathrm{d}s,$$

we can apply the arguments in the proof of [5, Lemma 3.5] and the first part of the proof of [5, Theorem 3.6] to get the conclusion of our proposition. □

2.2 Probabilistic representation of λ_2

The following result is a generalization of [22, Theorem 4.4.4] and it implies that (1.5) is valid when $\beta \in \mathbf{K}(\xi)$.

Proposition 2.2 (Probabilistic representation of λ_2) *Let*

$$\tau_n := \inf_{t\geq 0}\{t : \xi_t \notin D_n\}, \ n \geq 1.$$

Then

$$\lambda_2(\beta) = \sup_n \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D_n} \Pi_x(e_\beta(t); t < \tau_n).$$

Proof. Let $P_t^{\beta,n}$ stand for P_t^{β,D_n} and let

$$\lambda_2^n := \lim_{t\to\infty} \frac{1}{t} \log \|P_t^{\beta,n}\|_{2,2},$$

where $\|P_t^{\beta,n}\|_{2,2}$ stands for the operator norm of $P_t^{\beta,n}$ from $L^2(D_n, m)$ to $L^2(D_n, m)$. It is well known that

$$-\lambda_2(\beta) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d}(\nabla f a\nabla f)e^{2Q}\mathrm{d}x - \int_{\mathbb{R}^d} f^2\beta e^{2Q}\mathrm{d}x : f \in C_c^\infty(\mathbb{R}^d), \|f\|_2 = 1\right\} \quad (2.12)$$

and

$$-\lambda_2^n(\beta) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d}(\nabla f a\nabla f)e^{2Q}\mathrm{d}x - \int_{\mathbb{R}^d} f^2\beta e^{2Q}\mathrm{d}x : f \in C_c^\infty(D_n), \|f\|_2 = 1\right\}. \quad (2.13)$$

For any $n \geq 1$, by using (2.1)–(2.3) and Proposition 2.1 we can easily see that $\beta \in \mathbf{K}_\infty(\xi^{(n)})$ (The definition of the Kato class $\mathbf{K}_\infty(\xi^{(n)})$ is similar to Definition 1.6; see [4] for details.). Thus it follows from [3, Theorem 2.3] that for any $n \geq 1$,

$$-\lambda_2^n(\beta) = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in D_n} P_t^{\beta,n}1(x).$$

Now combining this with (2.12)–(2.13) yields the conclusion of our proposition. □

2.3 Properties of the gauge function

The following basic properties of g_β will be used later.

Lemma 2.3 *(1) For any open set $D \subset \mathbb{R}^d$ and nonnegative measurable function f on ∂D, if the function $g^D_{\beta,f}(x) := \Pi_x[e_\beta(\tau_D)f(\xi_{\tau_D})1_{\{\tau_D<\infty\}}]$ is not identically infinite on D, then for any compact set K, $g^D_{\beta,f}$ is bounded on K and there exists $A = A(D,K,\beta) > 1$, independent of f, such that*

$$\sup_{x\in K} g^D_{\beta,f}(x) \le A \inf_{x\in K} g^D_{\beta,f}(x). \tag{2.14}$$

Furthermore, $g^D_{\beta,f}$ is a continuous solution of $(L+\beta)h = 0$ in D in the sense of distributions.

(2) If g_β is not identically infinite in $\mathbb{R}^d$, then for any compact set $K \subset \mathbb{R}^d$, $g_\beta(x)$ is bounded on K and there exists an $A = A(K,\beta) > 1$ such that

$$\sup_{x\in K} g_\beta(x) \le A \inf_{x\in K} g_\beta(x). \tag{2.15}$$

Furthermore, $g_{\beta,f}$ is a continuous solution of $(L+\beta)h = 0$ in $\mathbb{R}^d$ in the sense of distributions.

(3) If g_β is not identically zero in $\mathbb{R}^d$, then $g_\beta(x) > 0$ for all $x \in \mathbb{R}^d$.

Proof. (1) The proof follows the same line of arguments as that of [5, Theorem 5.18]. Without loss of generality, we may and do assume that $K \subset B(0,n)$ and that there exists $x_1 \in K$ such that $g^D_{\beta,f}(x_1) < \infty$. Then by the definition of $g^D_{\beta,f}$ and the strong Markov property, for any ball $B = B(x_1,r) \subset \overline{B(x_1,r)} \subset D$, we have

$$g^D_{\beta,f}(x_1) = \Pi_{x_1}[e_\beta(\tau_B)g^D_{\beta,f}(\xi_{\tau_B})].$$

By (2.8)–(2.10) and Proposition 2.1, for any $\epsilon > 0$, we can choose $r_0 = r_0(n,\beta) \in (0,1]$ such that for any $r \in (0,r_0)$ and any $(x,z) \in B \times \partial B$:

$$\Pi^z_x \int_0^{\tau_B} e_{|\beta|}(t)\,\mathrm{d}t \le \frac{1}{2},$$

where Π^z_x stands for the law of the $M_B(\cdot,z)$-conditioned diffusion, i.e., the process such that for all bounded Borel function on B and $t > 0$,

$$\Pi^z_x[f(\xi_t)] = \frac{1}{M_B(x,z)}\Pi_x[f(\xi_t)M_B(\xi_t,z); t < \tau_B].$$

Repeating the argument of [5, Theorem 5.17], we get that

$$\frac{1}{2} \le \Pi^z_x e_\beta(\tau_B) \le 2.$$

Put $v(x,z) = \Pi_x^z e_\beta(\tau_B)$, then by [5, Proposition 5.12] (which is also valid for ξ by the same arguments contained in [5, Section 5.2]) we have

$$g^D_{\beta,f}(x_1) = \int_{\partial B} v(x_1,z)K_B(x_1,z)g^D_{\beta,f}(z)\sigma(\mathrm{d}z)$$

where σ stands for the surface measure on ∂B and K_B is the Poisson kernel of B with respect to ξ. It follows from the Harnack inequality (applied to the harmonic functions of ξ) that there exists some $c > 1$ such that

$$\sup_{x\in B(x_1,r/2)} K_B(x,z)) \leq c \inf_{x\in B(x_1,r/2)} K_B(x,z), \qquad \forall z \in \partial B.$$

Since, for $x \in B$,

$$g^D_{\beta,f}(x) = \int_{\partial B} v(x,z)K_B(x,z)g^D_{\beta,f}(z)\sigma(\mathrm{d}z),$$

therefore we have

$$\sup_{x\in B(x_1,r/2)} g^D_{\beta,f}(x) \leq c \inf_{x\in B(x_1,r/2)} g^D_{\beta,f}(x).$$

Now (2.14) follows from a standard chain argument. The last assertion of (1) can be proved by repeating the argument of the Corollary to [5, Theorem 5.18] and we omit the details.

(2) The proof of (2) is the same as that of (1).

(3) The proof of this part is the same as that [5, Proposition 8.10] and we omit the details. □

2.4 The operator G^β

For any $f \geq 0$ on $\mathbb{R}^d$, set

$$G^\beta f(x) := \Pi_x \int_0^\infty e_\beta(s)f(\xi_s)\,\mathrm{d}s. \tag{2.16}$$

$G^0 f$ will be denoted as Gf. The following result will be needed later.

Lemma 2.4 *Suppose that $f \geq 0$ is locally bounded on $\mathbb{R}^d$. If there exists an $x_1 \in \mathbb{R}^d$ such that $G^\beta f(x_1) < \infty$, then $G^\beta f$ is locally bounded on $\mathbb{R}^d$ as well.*

Proof. The proof is similar to that of the first part of Lemma 2.3. For convenience, we put $\widetilde{f} := G^\beta f$ in this proof. Without loss of generality, we may and do assume that the compact set K satisfies $K \subset B(0,n)$, and furthermore, that there exists an $x_1 \in K$

such that $\widetilde{f}(x_1) < \infty$. Let $v(x,z) := \Pi_x^z e_\beta(\tau_B)$. By the strong Markov property, for any $B = B(x_1, r)$, we have

$$\begin{aligned}
\widetilde{f}(x_1) &= \Pi_{x_1} \int_0^{\tau_B} e_\beta(s) f(\xi_s) \mathrm{d}s + \Pi_{x_1} \left[e_\beta(\tau_B) \Pi_{\xi_{\tau_B}} \int_0^\infty e_\beta(s) f(\xi_s) \mathrm{d}s \right] \\
&= \Pi_{x_1} \int_0^{\tau_B} e_\beta(s) f(\xi_s) \mathrm{d}s + \int_{\partial B} v(x_1, z) K_B(x_1, z) \widetilde{f}(z) \sigma(\mathrm{d}z).
\end{aligned} \tag{2.17}$$

By (2.8)–(2.10), Proposition 2.1 and the argument of [5, Theorem 5.17], for any $\epsilon > 0$, we can choose $r_0 = r_0(n, \beta) \in (0, 1]$ such that for any $r \in (0, r_0)$ and any $(x, z) \in B \times \partial B$:

$$\frac{1}{2} \le \Pi_x^z [e_\beta(\tau_B)] \le \Pi_x^z [e_{|\beta|}(\tau_B)] \le 2; \quad \Pi_x \tau_B^2 \le 2; \quad \Pi_x [e_{2|\beta|}(\tau_B)] \le 2.$$

Then we have

$$\widetilde{f}(x_1) \ge \frac{1}{2} \int_{\partial B} K_B(x_1, z) \widetilde{f}(z)) \sigma(\mathrm{d}z)$$

and

$$\begin{aligned}
\widetilde{f}(x) &= \Pi_x \int_0^{\tau_B} e_\beta(s) f(\xi_s) \mathrm{d}s + \int_{\partial B} v(x, z) K_B(x, z) \widetilde{f}(z) \sigma(\mathrm{d}z) \\
&\le C \Pi_x (\tau_B e_{|\beta|}(\tau_B)) + \int_{\partial B} v(x, z) K_B(x, z) \widetilde{f}(z) \sigma(\mathrm{d}z) \\
&\le C [\Pi_x \tau_B^2]^{1/2} [\Pi_x [e_{2|\beta|}(\tau_B)]]^{1/2} + \int_{\partial B} v(x, z) K_B(x, z) \widetilde{f}(z) \sigma(\mathrm{d}z),
\end{aligned}$$

where C is the upper bound of f on B. It follows from the Harnack inequality (for harmonic functions of ξ) that there exists some $c > 1$ such that

$$\sup_{x \in B(x_1, r/2)} K_B(x, z)) \le c \inf_{x \in B(x_1, r/2)} K_B(x, z).$$

Thus

$$\sup_{x \in B(x_1, r/2)} \widetilde{f}(x) \le 2C + 4c \widetilde{f}(x_1).$$

Now the assertion of the lemma follows from a standard chain argument. $\square$

3 Proofs of Theorem 1.6 and Theorem 1.7

For $\mathbf{0} \neq \mu \in M(\mathbb{R}^d)$, define

$$\Pi_\mu = \int_D \Pi_x \mu(\mathrm{d}x). \tag{3.1}$$

The following result is [7, Lemma 1.5].

Lemma 3.1 *We can rewrite the above equation* (1.7) *as*

$$u(t,x) + \Pi_x \int_0^t e_\beta(s) k(\xi_s)(u(t-s,\xi_s))^2 \mathrm{d}s = \Pi_x(e_\beta(t) f(\xi_t)). \tag{3.2}$$

Combining (1.6) and (3.2), we get the following expectation and variance formulas: for any bounded nonnegative function f on $\mathbb{R}^d$ and any nonzero $\mu \in M(\mathbb{R}^d)$,

$$\mathbb{P}_\mu \langle f, X_t \rangle = \Pi_\mu(f(\xi_t) e_\beta(t)) \tag{3.3}$$

and

$$\mathrm{Var}_\mu \langle f, X_t \rangle = \Pi_\mu \left(\int_0^t e_\beta(s) k(\xi_s) 2 [\Pi_{\xi_s} e_\beta(t-s) f(\xi_{t-s})]^2 \mathrm{d}s \right), \tag{3.4}$$

where Var_μ stands for variance under $\mathbb{P}_\mu$.

Lemma 3.2 *If* $\lambda_\infty > 0$ *then*

$$\liminf_{t\to\infty} \|P_t^\beta 1\|_\infty^{-1} \int_0^t \|P_s^\beta 1\|_\infty \,\mathrm{d}s < \infty. \tag{3.5}$$

Proof: For convenience, we denote $\|P_t^\beta 1\|_\infty$ by $h(t)$ in this proof. Suppose that the statement is false. Then

$$\lim_{t\to\infty} \frac{\int_0^t h(s)\,\mathrm{d}s}{h(t)} = \infty,$$

and so for any $K > 0$, there exists $T_K > 0$ such that for $t > T_K$,

$$\frac{\int_0^t h(s)\,\mathrm{d}s}{h(t)} > K,$$

i.e.,

$$h(t) < \frac{1}{K} \int_0^t h(s)\,\mathrm{d}s = \alpha + \frac{1}{K} \int_{T_K}^t h(s)\,\mathrm{d}s,$$

where $\alpha = \frac{1}{K} \int_0^{T_K} h(s)\,\mathrm{d}s$. By Gronwall's lemma, we get

$$h(t) \le \alpha \left(e^{(t-T_2)/K} - 1 \right).$$

However, if $1/K < \frac{\lambda_\infty}{2}$ $(K > \frac{2}{\lambda_\infty})$, this contradicts to the fact that

$$\lim_{t\to\infty} \frac{\log h(t)}{t} \ge \frac{\lambda_\infty}{2}.$$

This contradiction proves the lemma. $\square$

3.1 Proof of Theorem 1.6

For the proof of the theorem, we will need the following slight generalization of Doob's maximal inequality for submartingales.

Lemma 3.3 *Assume that $T \in (0,\infty)$, and that the non-negative, right continuous, filtered stochastic process $(M_t, \mathcal{F}_t, \mathbf{P})_{0\le t\le T}$ satisfies that there exists an $a > 0$ such that*

$$\mathbf{E}(M_t \mid \mathcal{F}_s) \ge aM_s,\ 0 \le s < t \le T.$$

Then, for every $\alpha \in (0,\infty)$ and $0 \le S \le T$,

$$\mathbf{P}\left(\sup_{t\in[0,S]} M_t \ge \alpha\right) \le (a\alpha)^{-1}\mathbf{E}[M_S].$$

Proof: Looking at the proof of Doob's inequality (see [27, Theorems 5.2.1 and 7.1.9] and their proofs), one can see that, when the submartingale property is replaced by our assumption, the whole proof goes through, except that now one has to include a factor a^{-1} on the right hand side. □

Proof of Theorem 1.6: (1) By a standard Borel-Cantelli argument, it suffices to prove that with an appropriate choice of $T > 0$, it is true that for any given $\epsilon > 0$,

$$\sum_n \mathbb{P}_\mu\left(\sup_{s\in[0,T]} e^{-\lambda(n+s)}\|X_{n+s}\| > \epsilon\right) < \infty. \tag{3.6}$$

Pick

$$\gamma \ge -\lambda. \tag{3.7}$$

Then

$$\mathbb{P}_\mu\left(\sup_{s\in[0,T]} e^{-\lambda(n+s)}\|X_{n+s}\| > \epsilon\right) \le \mathbb{P}_\mu\left(\sup_{s\in[0,T]} e^{\gamma(n+s)}\|X_{n+s}\| > \epsilon\cdot e^{(\lambda+\gamma)n}\right). \tag{3.8}$$

Let $M_t^{(n)} := e^{\gamma(n+t)}\|X_{n+t}\|$ for $t \in [0,T]$. Pick a number $0 < a < 1$ and fix it. Let $\mathcal{F}_s^{(n)} := \sigma(X_{n+r} : r \in [0,s])$. If we show that for a sufficiently small $T > 0$ and all $n \ge 1$, the process $\{M_t^{(n)}\}_{0\le t\le T}$ satisfies that for all $0 < s < t < T$,

$$\mathbb{P}_\mu(M_t^{(n)} \mid \mathcal{F}_s^{(n)}) \ge aM_s^{(n)}, \tag{3.9}$$

then, by using Lemma 3.3, we can continue (3.8) with

$$\begin{aligned}\mathbb{P}_\mu\left(\sup_{s\in[0,T]} e^{-\lambda(n+s)}\|X_{n+s}\| > \epsilon\right) &\leq \frac{1}{a\epsilon}e^{-(\lambda+\gamma)n}\mathbb{P}_\mu\left[e^{\gamma(n+T)}\|X_{n+T}\|\right] \\ &= \frac{1}{a\epsilon}e^{(\lambda+\gamma)T}e^{-\lambda(n+T)}\mathbb{P}_\mu\|X_{n+T}\| \\ &\leq \frac{\|\mu\|}{a\epsilon}e^{(\lambda+\gamma)T}e^{-\lambda(n+T)}\|P^\beta_{n+T}1\|_\infty.\end{aligned}$$

Since $\lambda > \lambda_\infty$ and $\|P^\beta_{n+T}1\|_\infty = \exp(\lambda_\infty(n+T)+o(n))$ as $n\to\infty$, therefore (3.6) holds.

It remains to check (3.9). Let $0 < s < t < T$. Using the Markov and branching properties at time $n+s$,

$$\begin{aligned}\mathbb{P}_\mu\left[M^{(n)}_t \mid \mathcal{F}^{(n)}_s\right] &= \mathbb{P}_{X_{n+s}}e^{\gamma(n+t)}\|X_{t-s}\| = \left\langle \mathbb{P}_{\delta_x}e^{\gamma(n+t)}\|X_{t-s}\|,\ X_{n+s}(\mathrm{d}x)\right\rangle \\ &= \left\langle \mathbb{P}_{\delta_x}e^{\gamma(t-s)}\|X_{t-s}\|,\ e^{\gamma(n+s)}X_{n+s}(\mathrm{d}x)\right\rangle. \end{aligned}\tag{3.10}$$

At this point we are going to determine T as follows. According to the assumption $\beta\in\mathbf{K}(\xi)$,

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\Pi_x\int_0^t |\beta|(\xi_s)\,\mathrm{d}s = 0.$$

Pick $T>0$ such that

$$\gamma t + \Pi_x\int_0^t \beta(\xi_s)\,\mathrm{d}s \geq \log a,$$

for all $0<t<T$ and all $x\in\mathbb{R}^d$. By Jensen's inequality,

$$\gamma t + \log \Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right) \geq \log a,$$

and thus

$$\mathbb{P}_{\delta_x}e^{\gamma t}\|X_t\| = e^{\gamma t}\Pi_x\exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right) \geq a$$

holds too, for all $0<t<T$ and all $x\in\mathbb{R}^d$. Returning to (3.10), for $0<s<t<T$,

$$\mathbb{P}_\mu[M^{(n)}_t \mid \mathcal{F}^{(n)}_s] \geq \left\langle a,\ e^{\gamma(n+s)}X_{n+s}(\mathrm{d}x)\right\rangle = aM^{(n)}_s,$$

yielding (3.9).

(2) First note that to prove (1.12) it suffices to prove that there exists $c_0>0$ such that for all $K>0$,

$$\mathbb{P}_\mu(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| \geq K) \geq c_0. \tag{3.11}$$

Since

$$\{\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| \geq K\} \supseteq \limsup_{t\to\infty}\{e^{-\lambda t}\|X_t\| \geq K\},$$

we have by Fatou's lemma,

$$\begin{aligned}\mathbb{P}_\mu(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| \geq K) &\geq \limsup_{t\to\infty} \mathbb{P}_\mu(e^{-\lambda t}\|X_t\| \geq K) \\ &= \limsup_{t\to\infty} \mathbb{P}_\mu(e^{-\lambda t}\|X_t\| - K \geq 0).\end{aligned} \tag{3.12}$$

The assumption $\lambda < \lambda_\infty$ implies that

$$\lim_{t\to\infty} \mathbb{P}_\mu(e^{-\lambda t}\|X_t\|) = \lim_{t\to\infty} e^{-\lambda t}\Pi_\mu e_\beta(t) = \infty. \tag{3.13}$$

Thus $\mathbb{P}_\mu e^{-\lambda t}\|X_t\| > K$ for large t. Applying the inequality in [6, Ex. 1.3.8] (with $Y = e^{-\lambda t}\|X_t\|$ and $a = K$), we get

$$\mathbb{P}_\mu(e^{-\lambda t}\|X_t\| - K \geq 0) \geq \frac{\left(\mathbb{P}_\mu e^{-\lambda t}\|X_t\| - K\right)^2}{\mathbb{P}_\mu(e^{-\lambda t}\|X_t\|)^2}. \tag{3.14}$$

By (3.3) and (3.4), (3.12) and (3.14) yield

$$\begin{aligned} &\mathbb{P}_\mu(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| \geq K) \\ \geq\ & \limsup_{t\to\infty} \frac{(\Pi_\mu e^{-\lambda t} e_\beta(t) - K)^2}{(\Pi_\mu e^{-\lambda t} e_\beta(t))^2 + 2e^{-2\lambda t}\Pi_\mu \displaystyle\int_0^t e_\beta(s)k(\xi_s)[\Pi_{\xi_s} e_\beta(t-s)]^2 \,\mathrm{d}s} \\ =\ & \limsup_{t\to\infty} \left(1 - K\frac{e^{\lambda t}}{\Pi_\mu e_\beta(t)}\right)^2 \left(1 + 2\frac{\Pi_\mu\left(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s} e_\beta(t-s)\,\mathrm{d}s\right)}{(\Pi_\mu e_\beta(t))^2}\right)^{-1} \\ =\ & \limsup_{t\to\infty} \left(1 + 2\frac{\Pi_\mu\left(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s} e_\beta(t-s)\,\mathrm{d}s\right)}{(\Pi_\mu e_\beta(t))^2}\right)^{-1}. \end{aligned} \tag{3.15}$$

Note that

$$\Pi_{\xi_s} e_\beta(t-s) \leq \|P^\beta_{(t-s)} 1\|_\infty.$$

Thus we have

$$\begin{aligned}\Pi_\mu\left(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s} e_\beta(t-s)\,\mathrm{d}s\right) \leq\ & \|k\|_\infty \Pi_\mu e_\beta(t)\left[\int_0^t \|P^\beta_{t-s}1\|_\infty \,\mathrm{d}s\right] \\ =\ & \|k\|_\infty \Pi_\mu e_\beta(t)\left[\int_0^t \|P^\beta_s 1\|_\infty \,\mathrm{d}s\right].\end{aligned}$$

So, we have for every $K > 0$,

$$\mathbb{P}_\mu(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| \geq K) \geq \left(1 + 2\liminf_{t\to\infty} \frac{\|k\|_\infty \|P^\beta_t 1\|^{-1}_\infty \int_0^t \|P^\beta_s 1\|_\infty \,\mathrm{d}s}{\|P^\beta_t 1\|^{-1}_\infty \Pi_\mu e_\beta(t)}\right)^{-1}. \tag{3.16}$$

By Lemma 3.2,

$$\liminf_{t\to\infty} \|P_t^\beta 1\|_\infty^{-1} \int_0^t \|P_s^\beta 1\|_\infty \, \mathrm{d}s < \infty. \tag{3.17}$$

By Fatou's lemma and (1.11),

$$\liminf_{t\to\infty} \|P_t^\beta 1\|_\infty^{-1} \Pi_\mu e_\beta(t) \geq \langle \mu, \liminf_{t\to\infty} \|P_t^\beta 1\|_\infty^{-1} \Pi_\cdot e_\beta(t) \rangle > 0. \tag{3.18}$$

Combining (3.18) and Lemma 3.2, we arrive at (3.11). $\square$

3.2 Proof of Theorem 1.7

(1) Using Fatou's lemma, we get

$$\liminf_{t\to\infty} e^{-\lambda_\infty t} \Pi_\mu e_\beta(t) = \liminf_{t\to\infty} \Pi_\mu e_{\beta-\lambda_\infty}(t) \geq \left\langle \liminf_{t\to\infty} \Pi_\cdot e_{\beta-\lambda_\infty}(t), \mu \right\rangle = \infty,$$

which means that (3.13) holds with λ replaced by λ_∞. So the proof of Theorem 1.6(2) works with λ replaced by λ_∞.

(2) By (3.3), we have

$$\mathbb{P}_\mu[\exp(-\lambda_\infty t)\|X_t\|] = \Pi_\mu e_{\beta-\lambda_\infty}(t). \tag{3.19}$$

Letting $t \to \infty$ and using Fatou's lemma, we get

$$\mathbb{P}_\mu(\liminf_{t\to\infty} \exp(-\lambda_\infty t)\|X_t\|) \leq \liminf_{t\to\infty} \Pi_\mu e_{\beta-\lambda_\infty}(t). \tag{3.20}$$

Note that $\Pi_\mu e_{\beta-\lambda_\infty}(t) = \langle \Pi_\cdot e_{\beta-\lambda_\infty}(t), \mu \rangle$. Using (1.15) we get

$$\lim_{t\to\infty} \Pi_\mu e_{\beta-\lambda_\infty}(t) = \langle \lim_{t\to\infty} \Pi_\cdot e_{\beta-\lambda_\infty}(t), \mu \rangle = \langle g_{\beta-\lambda_\infty}, \mu \rangle = 0,$$

where in the first equality we used the inequality $\Pi_\cdot e_{\beta-\lambda_\infty}(t) \leq \sup_{x\in\mathbb{R}^d} \Pi_x(\sup_{t\geq 0} e_{\beta-\lambda_\infty}(t))$ and the fact that μ is finite measure, and in the second equality we used the inequality $e_{\beta-\lambda_\infty}(t) \leq \sup_{t\geq 0} e_{\beta-\lambda_\infty}(t) < \infty$ Π_x-a.s. for any $x \in \mathbb{R}^d$, and (1.15). Hence by (3.20) we get

$$\mathbb{P}_\mu\left(\liminf_{t\to\infty} \exp(-\lambda_\infty t)\|X_t\| = 0\right) = 1,$$

which implies (1.16).

Finally, when $\beta \leq 0$, trivially $\lambda_\infty \leq 0$; hence $\mathbb{P}_\mu(\liminf_{t\to\infty} \|X_t\| = 0) = 1$. On the other hand, $\|X\|$ is a supermartingale by the expectation formula and the branching Markov property, and thus, $\lim_{t\to\infty} \|X_t\|$ exists $\mathbb{P}_\mu$-a.s. Hence, we can improve the liminf to a limit. $\square$

4 Proofs of Theorems 1.8 and 1.9

4.1 A 'spine' proof of Theorem 1.8

We start with a lemma.

Lemma 4.1 *Assume that $\beta \in \mathbf{K}(\xi)$ and that $h > 0$ is a bounded solution to*

$$(L + \beta - \lambda_\infty)h = 0 \text{ in } \mathbb{R}^d$$

in the sense of distributions. Let $\mathbf{0} \neq \mu \in M(\mathbb{R}^d)$ and $\mathcal{F}_t := \sigma\{X_r, r \leq t\}$. Then the process $(e_{-\lambda_\infty}(t)\langle h, X_t\rangle; \mathcal{F}_t)_{t\geq 0}$ is a positive $\mathbb{P}_\mu$-martingale.

Proof. Recall that $D_n = B(0, n)$ and τ_n is the first exit time of ξ from D_n. Since h is harmonic with respect to the operator $L + \beta - \lambda_\infty$, we have

$$h(x) = \Pi_x \left[e_{\beta-\lambda_\infty}(t \wedge \tau_n)h(\xi_{t\wedge\tau_n})\right], \quad \text{for every } n \geq 1 \text{ and } t \geq 0, \tag{4.1}$$

see the proof of [26, Lemma 2.1]. Since h is bounded, bounded convergence yields

$$h(x) = \Pi_x \left[e_{\beta-\lambda_\infty}(t)h(\xi_t)\right], \quad \text{for every } t \geq 0. \tag{4.2}$$

By the branching and Markov properties, for $r \leq s < t$, we have

$$\begin{aligned}
&\mathbb{P}_\mu(e_{-\lambda_\infty}(t)\langle h, X_t\rangle|\mathcal{F}_s) \\
= \;& e_{-\lambda_\infty}(t)\mathbb{P}_{X_s}\langle h, X_{t-s}\rangle \\
= \;& e_{-\lambda_\infty}(t)\langle \Pi_\cdot \left(e_\beta(t-s)h(\xi_{t-s})\right), X_s\rangle \\
= \;& e_{-\lambda_\infty}(t)\langle \Pi_\cdot \left(e_\beta(t-s)h(\xi_{t-s})\right), X_s\rangle \\
= \;& e_{-\lambda_\infty}(s)\langle h, X_s\rangle,
\end{aligned} \tag{4.3}$$

which means that $e_{-\lambda_\infty}(t)\langle h, X_t\rangle$ is a martingale under $\mathbb{P}_\mu$. $\square$

Since M^h defined by

$$M^h_t := \exp(-\lambda_\infty t)\langle h, X_t\rangle$$

is a nonnegative $\mathbb{P}_\mu$-martingale, therefore $\lim_{t\to\infty} M^h_t$ exists $\mathbb{P}_\mu$-a.s. It follows from (4.4) and Lemma 2.4 that

$$\Pi_\mu \int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)\mathrm{d}s < \infty.$$

Define $\widetilde{\mathbb{P}}_\mu$ by the martingale change of measure

$$\left.\frac{d\widetilde{\mathbb{P}}_\mu}{d\mathbb{P}_\mu}\right|_{\mathcal{F}_t} = \frac{1}{\langle h, \mu\rangle}M^h_t.$$

Following [8] we make the following observations. First, the probability measure $\widetilde{\mathbb{P}}_\mu$ corresponds to the so-called 'spine-decomposition' of the process. Secondly, M^h is a positive $\widetilde{\mathbb{P}}_\mu$-supermartingale, and thus it has a $\widetilde{\mathbb{P}}_\mu$-a.s. limit. Finally, by [8, Theorem 5(ii)],

$$\begin{aligned}\widetilde{E}_\mu(M_t^\phi) = &\ \langle h,\mu\rangle + \Pi_{h\mu}^h\left(\int_0^t e^{-\lambda_\infty s}2(kh)(\xi_s)\mathrm{d}s\right)\\ = &\ \langle h,\mu\rangle + 2(\langle h,\mu\rangle)^{-1}\Pi_\mu\left(\int_0^t e^{-2\lambda_\infty s+\beta}(kh^2)(\xi_s)\mathrm{d}s\right),\end{aligned}$$

where

$$\left.\frac{d\Pi_x^h}{d\Pi_x}\right|_{\mathcal{G}_t} = \frac{h(\xi_t)}{h(x)}\exp\left(\int_0^t(-\lambda_\infty+\beta(\xi_s))\mathrm{d}s\right),$$

where $\mathcal{G}_t = \sigma(\xi_s : s\in[0,t])$. (In [8], the domain was bounded and instead of h, the unique solution to the Dirichlet boundary value problem was considered. However, the only property of the solution that was used in the proof was its invariance under the Feynman-Kac semigroup. We have this property for our h too by (4.2).)

After the observations above, the rest is just standard measure theory. Namely, by Fatou's lemma,

$$\widetilde{E}_\mu(M_\infty^\phi) \le \lim_{t\to\infty}\widetilde{E}_\mu(M_t^h) < \infty.$$

Consequently, $\lim_{t\to\infty} M_t^h < \infty$, $\widetilde{\mathbb{P}}_\mu$-a.s., and therefore M_t^h converges in $L^1(\mathbb{P}_\mu)$.

Finally, suppose that, with probability one, $\|X_t\| = 0$ for some $t > 0$. Since convergence in mean implies that $\lim_{t\to\infty} M_t^h$ is not identically zero, we get a contradiction. □

4.2 Alternative proof of Theorem 1.8

The following result will imply Theorem 1.8 very easily.

Theorem 4.2 *Suppose that $\beta\in\mathbf{K}(\xi)$ and that $h>0$ is a bounded solution $(L+\beta-\lambda_\infty)h = 0$ in $\mathbb{R}^d$ in the sense of distributions. If there exists an $x_0\in\mathbb{R}^d$ such that*

$$\Pi_{x_0}\left[\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s)\,\mathrm{d}s\right] < \infty, \tag{4.4}$$

then for $\mathbf{0}\neq\mu\in M_c(\mathbb{R}^d)$, one has $\lim_{t\to\infty}\exp(-\lambda_\infty t)\langle h,X_t\rangle$ exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$, and

$$0 < \mathbb{P}_\mu\left[\lim_{t\to\infty}\exp(-\lambda_\infty t)\langle h,X_t\rangle\right]^2 < \infty,$$

which implies that

$$\mathbb{P}_\mu\left(\lim_{t\to\infty}\exp(-\lambda_\infty t)\langle h,X_t\rangle = 0\right) < 1 \tag{4.5}$$

and

$$\mathbb{P}_\mu\left(\lim_{t\to\infty}\exp(-\lambda_\infty t)\langle h,X_t\rangle = \infty\right) = 0. \tag{4.6}$$

Indeed, Theorem 1.8 immediately follows from this result, because under (1.18), (4.4) is equivalent to (1.17), and (4.5) and (4.6) imply (1.20) and (1.19), respectively. The assertions of Theorem 1.8 follow easily.

It remains to prove Theorem 4.2.

Proof. We have seen that the martingale limit $\lim_{t\to\infty} M_t^h$ exists $\mathbb{P}_\mu$-a.s. By the martingale property, we have

$$\mathbb{P}_\mu M_t^h = \exp(-\lambda_\infty t)\Pi_\mu[e_\beta(t)h(\xi_t)] = \langle h, \mu\rangle.$$

It follows from (4.4) and Lemma 2.4 that

$$\Pi_\mu\left[\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s))\,\mathrm{d}s\right] < \infty.$$

Thus by the variance formula (3.4) and (4.1), we have

$$\begin{aligned}
&\mathbb{P}_\mu\left[M_t^h\right]^2\\
=\ &\langle h,\mu\rangle^2 + \exp(-2\lambda_\infty t)\Pi_\mu\left[\int_0^t e_\beta(s)k(\xi_s)[\Pi_{\xi_s}(e_\beta(t-s)h(\xi_{t-s}))]^2\,\mathrm{d}s\right]\\
=\ &\langle h,\mu\rangle^2 + \Pi_\mu\left[\int_0^t e_\beta(s)\exp(-2\lambda_\infty s)k(\xi_s)[\Pi_{\xi_s}(e_{\beta-\lambda_\infty}(t-s)h(\xi_{t-s}))]^2\,\mathrm{d}s\right]\\
=\ &\langle h,\mu\rangle^2 + \Pi_\mu\left[\int_0^t e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s))\,\mathrm{d}s\right].
\end{aligned}$$

By the L^2-convergence theorem, M_t^h converges to some η in $L^2(\mathbb{P}_\mu)$. In particular,

$$0 < \mathbb{P}_\mu\eta^2 = \langle h,\mu\rangle^2 + \Pi_\mu\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s))\,\mathrm{d}s < \infty,$$

and therefore,

$$\mathbb{P}_\mu(\eta<\infty) = 1, \quad \text{and} \quad \mathbb{P}_\mu(\eta = 0) < 1. \qquad \square$$

4.3 Preparation for the proof of Theorem 1.9

In the remainder of this section, we suppose $\lambda_\infty = 0$ and that $h > 0$ is a bounded solution to $(L+\beta)u = 0$ in $\mathbb{R}^d$ in the sense of distributions. For $c > 0$, put

$$u_{ch}(t,x) := -\log\mathbb{P}_{\delta_x}\exp(-c\langle h, X_t\rangle), \tag{4.7}$$

then $u_{ch}(t,x)$ is a solution of the following integral equation:

$$u_{ch}(t,x) + \Pi_x\int_0^t \left[k(\xi_r)\left(u_{ch}(t-r,\xi_r)\right)^2 - \beta(\xi_r)u_{ch}(t-r,\xi_r)\right]\mathrm{d}r = c\Pi_x h(\xi_t). \tag{4.8}$$

By Lemma 3.1, the above integral equation is equivalent to

$$u_{ch}(t,x) + \Pi_x \int_0^t e_\beta(r)k(\xi_r)\left[u_{ch}(t-r,\xi_r)\right]^2 \mathrm{d}r = c\Pi_x\left[e_\beta(t)h(\xi_t)\right]. \tag{4.9}$$

Since h is a bounded positive solution to $(L+\beta)u=0$, we have

$$\Pi_x\left[e_\beta(t)h(\xi_t)\right] = h(x).$$

Thus (4.9) can be rewritten as

$$u_{ch}(t,x) + \Pi_x\left[\int_0^t e_\beta(r)k(\xi_r)\left[u_{ch}(t-r,\xi_r)\right]^2 \mathrm{d}r\right] = ch(x). \tag{4.10}$$

In particular,

$$u_{ch}(t,x) \le ch(x). \tag{4.11}$$

Put

$$u_{ch}(x) := -\log \mathbb{P}_{\delta_x} \exp(-c\lim_{t\to\infty}\langle h, X_t\rangle). \tag{4.12}$$

By Lemma 4.1, under $\mathbb{P}_\mu$, $\exp(-c\langle h, X_t\rangle)$, $t\ge 0$ is a bounded submartingale. Thus $u_{ch}(t,x)$ is non-increasing in t. Hence, by the dominated convergence theorem, for every $x\in\mathbb{R}^d$,

$$u_{ch}(t,x)\downarrow u_{ch}(x) \quad \text{ as } t\uparrow\infty.$$

Note that if k and β are radial functions, and if L is radial, then $u_{ch}(\cdot)$ is a radial function, i.e.,

$$u_{ch}(x) = u_{ch}(\|x\|).$$

Lemma 4.3 *(1) For any $x\in\mathbb{R}^d$ and $r>0$,*

$$u_{ch}(x) \le \Pi_x(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)})).$$

(2) If L, k and β are radial, then

$$u_{ch}(x) = u_{ch}(\|x\|) \le u_{ch}(R)\Pi_x(e_\beta(\tau_{B(0,R)})), \quad \|x\|<R. \tag{4.13}$$

Proof. (1) By the special Markov property, for every fixed $x\in\mathbb{R}^d$, one has

$$\begin{aligned}\exp(-u_{ch}(x)) =& \ \mathbb{P}_{\delta_x}\exp(-c\lim\nolimits_{t\to\infty}\langle h, X_t\rangle)\\ =& \ \mathbb{P}_{\delta_x}\Big(P_{X_{\tau_{B(x,r)}}}\exp(-c\lim\nolimits_{t\to\infty}\langle h, X_t\rangle)\Big)\\ =& \ \mathbb{P}_{\delta_x}\exp\langle -u_{ch}, X_{\tau_{B(x,r)}}\rangle.\end{aligned}$$

By Jensen's inequality,

$$\exp(-u_{ch}(x)) \geq \exp(-\mathbb{P}_{\delta_x}\langle u_{ch}, X_{\tau_{B(x,r)}}\rangle) = \exp[-\Pi_x(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)}))],$$

which implies $u_{ch}(x) \leq \Pi_x(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)}))$.

(2) Similarly we have, for $x \in B(0,R)$, that

$$u_{ch}(x) \leq u_{ch}(R)\Pi_x(e_\beta(\tau_{B(0,R)})).$$

□

Note that $u_{ch}(x)$ is increasing in c. Let

$$u_{ch}(x) \uparrow u_\infty(x) = -\log \mathbb{P}_{\delta_x}(\lim_{t\to\infty}\langle h, X_t\rangle = 0). \tag{4.14}$$

Lemma 4.4 *Either $u_\infty(x) \equiv 0$ or $u_\infty \in (0,\infty]$ in $\mathbb{R}^d$. That is, if*

$$E_h := \left\{\lim_{t\to\infty}\langle h, X_t\rangle = 0\right\},$$

then either $\mathbb{P}_{\delta_x}(E_h) = 1, \forall x \in \mathbb{R}^d$, or $\mathbb{P}_{\delta_x}(E_h) < 1, \forall x \in \mathbb{R}^d$.

Proof. We first prove that if there exists a measurable set $A \subset \mathbb{R}^d$ with positive Lebesgue measure such that $u_\infty > 0$ on A, then $u_\infty(x) > 0$ for every $x \in \mathbb{R}^d$. Indeed, for every $x \in \mathbb{R}^d$,

$$\begin{aligned} & \mathbb{P}_{\delta_x}(\lim_{t\to\infty}\langle h, X_t\rangle = 0) \\ = \ & \mathbb{P}_{\delta_x}(\mathbb{P}_{X(1)}(\lim_{t\to\infty}\langle h, X_t\rangle = 0)) \\ = \ & \mathbb{P}_{\delta_x}\exp\langle -u_\infty, X(1)\rangle. \end{aligned} \tag{4.15}$$

Note that

$$\mathbb{P}_{\delta_x}\langle u_\infty, X(1)\rangle = \Pi_x(u_\infty(\xi_1)e_\beta(1)) > 0. \tag{4.16}$$

(4.15) implies that $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\langle h, X_t\rangle = 0) < 1$. Thus we have $u_\infty(x) > 0$.

Now we prove that if $u_\infty = 0$ almost everywhere, then $u_\infty \equiv 0$. By (4.16), we know that $\mathbb{P}_{\delta_x}\langle u_\infty, X(1)\rangle = 0$, and thus $\langle u_\infty, X(1)\rangle = 0$, $\mathbb{P}_{\delta_x}$-a.s. By (4.15),

$$\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\langle h, X_t\rangle = 0) = 1.$$

Hence $u_\infty(x) = 0$ for every $x \in \mathbb{R}^d$. □

4.4 Proof of Theorem 1.9

Since $\beta \in \mathbf{K}_\infty(\xi)$, by the Gauge Theorem (see [4, Theorem 2.2] or [2, Theorem 2.6]), the assumption that g_β is not identically infinite implies that g_β is bounded between two positive numbers. By [2, Corollary 2.16], we have

$$\Pi_x\left[\sup_{0\le t\le\infty} e_\beta(t)\right] < \infty, \quad \forall x \in \mathbb{R}^d.$$

By dominated convergence,

$$g_\beta(x) = \lim_{R\to\infty} \Pi_{0,x}(e_\beta(\tau_{B(0,R)})), \quad x \in \mathbb{R}^d.$$

Take $h = g_\beta$. We know that h is a bounded solution of $(L+\beta)u = 0$ and satisfies (1.18); by Lemma 4.4 we only need to prove that if for every $x \in \mathbb{R}^d$, $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\|X_t\| = 0) < 1$, then

$$\Pi_x \int_0^\infty e_\beta(s)k(\xi_s)\,\mathrm{d}s < \infty, \quad x \in \mathbb{R}^d. \tag{4.17}$$

First note that the assumption that $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\|X_t\| = 0) < 1, x \in \mathbb{R}^d$ implies that $u_{ch}(x) = -\log \mathbb{P}_{\delta_x}\exp(-c\lim_{t\to\infty}\langle h, X_t\rangle) > 0$ for every $x \in \mathbb{R}^d$.

Since $u_{ch}(s,x) \ge u_{ch}(x)$ for every $s \in [0,t]$ and $x \in \mathbb{R}^d$, by (4.10), we have

$$\Pi_x \int_0^t e_\beta(s)k(\xi_s)u_{ch}^2(\xi_s)\,\mathrm{d}s \le ch(x), \quad x \in \mathbb{R}^d.$$

Letting $t \to \infty$, we get

$$\Pi_x \int_0^\infty e_\beta(s)k(\xi_s)u_{ch}^2(\xi_s)\,\mathrm{d}s \le ch(x), \quad x \in \mathbb{R}^d,$$

which can be rewritten as

$$\int_{\mathbb{R}^d} G_\beta(x,y)k(y)u_{ch}^2(y)m(\mathrm{d}y) \le ch(x), \quad x \in \mathbb{R}^d. \tag{4.18}$$

Letting $R \to \infty$ in (4.13), one gets

$$u_{ch}(x) \le h(x)\liminf_{R\to\infty} u_{ch}(R).$$

Since $u_{ch}(x) > 0$ and $0 < h(x) < \infty$, we have $\liminf_{R\to\infty} u_{ch}(R) > 0$. Then (4.18) implies (4.17). $\square$

5 Examples

5.1 Some super-diffusions with $\lambda_\infty > \lambda_2$

We start with an example in one dimension and with constant mass creation.

Example 5.1 Consider the elliptic operator

$$L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - b_0\frac{\mathrm{d}}{\mathrm{d}x}$$

on $\mathbb{R}$, where $b_0 > 0$ is a constant. Then the diffusion corresponding to L is conservative and transient. It is easy to see that the corresponding generalized principal eigenvalue is $\lambda_2(\mathbf{0}) = -b_0^2/2$. Let the potential β be a nonnegative constant. We have $\lambda_2(\beta) = \beta - b_0^2/2$ and $\lambda_\infty(\beta) = \beta$. The Green function of ξ is $G(x,y) = \frac{2\pi}{b_0}\exp\left(-2b_0(x-y)^+\right)$. Note that $L - \beta + \lambda_\infty(\beta) = L$.

For the large time behavior of X the following hold.

(i) According to [23, Theorem 7 and Example 1], X exhibits local extinction if and only if $\beta \in [0, b_0^2/2]$. Furthermore, when $\beta \in (b_0^2/2, \infty)$, X does not exhibit local extinction, and the exponential expected growth rate of the local mass is $(\beta - b_0^2/2)$. More precisely, for any continuous function g on $\mathbb{R}$ with compact support and any $\mathbf{0} \neq \mu \in M_c(\mathbb{R})$, one has

$$\lim_{t\to\infty} e^{\varrho t}\mathbb{P}_\mu\langle g, X_t\rangle = \begin{cases} 0, & \varrho \le \beta - b_0^2/2, \\ +\infty, & \varrho > \beta - b_0^2/2. \end{cases}$$

In fact, by [8], the local mass grows exponentially with positive probability, that is, not just in expectation.

(ii) If $\beta > 0$, since $\Pi_x e_\beta(t) = e^{\beta t}$ for all $x \in \mathbb{R}$ and $t \ge 0$, (1.11) is satisfied. Thus by Theorem 1.6, we have that, for any $\lambda > \beta$,

$$\mathbb{P}_\mu\left(\liminf_{t\to\infty} e^{-\lambda t}\|X_t\| = 0\right) = 1,$$

and that if k is bounded, then, for any $\lambda < \beta$,

$$\mathbb{P}_\mu\left(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| = \infty\right) > 0.$$

(iii) Since $u \equiv 1$ solves $Lu = 0$, by Theorem 4.2, if there exists an $x_0 \in \mathbb{R}$ such that

$$\Pi_{x_0}\int_0^\infty e^{-\beta s}k(\xi_s)\,\mathrm{d}s < \infty, \tag{5.1}$$

then for $\mathbf{0} \neq \mu \in M_c(\mathbb{R}^d)$, the limit $\lim_{t\to\infty}\exp(-\beta t)\|X_t\|$ exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$, and

$$0 < \mathbb{P}_\mu\left[\lim_{t\to\infty}\exp(-\beta t)\|X_t\|\right]^2 < \infty.$$

Hence,

$$\mathbb{P}_\mu(\lim_{t\to\infty} \exp(-\beta t)\|X_t\| = 0) < 1,$$

and

$$\mathbb{P}_\mu(\lim_{t\to\infty} \exp(-\beta t)\|X_t\| = \infty) = 0.$$

(iv) Since L is radial, by Theorem 1.9 we have that in the case of critical branching ($\beta = 0$), if

$$\int_{-\infty}^{x} \exp\left(-b_0(x-y)\right) k(y)\mathrm{d}y + \int_{x}^{\infty} k(y)\mathrm{d}y = \infty, \quad x \in \mathbb{R}, \tag{5.2}$$

then

$$\mathbb{P}_\mu\left(\lim_{t\to\infty} \|X_t\| = 0\right) = 1.$$

In summary,

(a) If $\beta > 0$, the exponential growth rate of the total mass is β.

(b) If $\beta = 0$, weak extinction depends on the branching rate function k: the superprocess exhibits weak extinction if and only if (5.2) holds.

In the next example the motion component is a multidimensional 'outward Ornstein-Uhlenbeck' process.

Example 5.2 Consider the elliptic operator

$$L = \frac{1}{2}\Delta + \gamma x \cdot \nabla \quad \text{on } \mathbb{R}^d,$$

where $d \geq 1$ and $\gamma > 0$. Then the diffusion corresponding to L is conservative and transient, and $\lambda_2(\mathbf{0}) = -\gamma d$. Let the potential β be a positive constant. Then $\lambda_2(\beta) = \beta - \gamma d$, and $\lambda_\infty(\beta) = \beta$.

(i) X exhibits local extinction if and only if $\beta \in [0, \gamma d]$. If $\beta \in (\gamma d, \infty)$, X does not exhibit local extinction, and the exponential growth rate of the local mass is $\beta - \gamma d$. More precisely, for any continuous function g on $\mathbb{R}^d$ with compact support,

$$\lim_{t\to\infty} e^{(\beta-\gamma d)t}\langle g, X_t\rangle = N_\mu \int_{\mathbb{R}^d} g(x)\exp(-\gamma|x|^2/2)\mathrm{d}x, \quad \text{in } \mathbb{P}_\mu\text{–probability}$$

for some random variable N_μ with mean $\int_{\mathbb{R}^d} \exp(-\gamma|x|^2/2)\mu(\mathrm{d}x)$ whenever

$$k(x) \leq K\exp(\gamma|x|^2/2), \quad K > 0,$$

and the starting measure $\mu = X_0$ satisfies

$$\int_{\mathbb{R}^d} \exp(-\gamma|x|^2/2)\mu(\mathrm{d}x) < \infty.$$

See [11, Theorem 1] and [10, Example 23].

(ii) By Theorem 1.6, we have that, for any $\lambda > \beta$,

$$\mathbb{P}_\mu(\liminf_{t\to\infty} e^{-\lambda t}\|X_t\| = 0) = 1,$$

and that if k is bounded in $\mathbb{R}^d$, then, for any $\lambda < \beta$,

$$\mathbb{P}_\mu(\limsup_{t\to\infty} e^{-\lambda t}\|X_t\| = \infty) > 0.$$

(iii) Obviously, $u \equiv 1$ is a bounded solution to $Lu = 0$, and using Theorem 4.2, we have that if the branching rate k satisfies

$$\Pi_x \int_0^\infty e^{-\beta s} k(\xi_s)\, \mathrm{d}s < \infty, \quad x \in \mathbb{R}^d,$$

then for $\mathbf{0} \neq \mu \in M_c(\mathbb{R}^d)$, there exists $\lim_{t\to\infty} \exp(-\beta t)\|X_t\|$ $\mathbb{P}_\mu$-a.s., and

$$\mathbb{P}_\mu \left[\lim_{t\to\infty} \exp(-\beta t)\|X_t\|\right]^2 \in (0, \infty).$$

Hence,

$$\mathbb{P}_\mu(\lim_{t\to\infty} \exp(-\beta t)\|X_t\| = 0) < 1,$$

and

$$\mathbb{P}_\mu(\lim_{t\to\infty} \exp(-\beta t)\|X_t\| = \infty) = 0.$$

5.2 Extinction and weak extinction

Next is an example illustrating the difference between extinction and weak extinction. The superprocess X below exhibits local extinction and also weak extinction, nevertheless it survives with positive probability.

Example 5.3 (Weak and also local extinction, but survival) Let $B, \epsilon >$ and consider the super-Brownian motion in $\mathbb{R}$ with $\beta(x) \equiv -B$ and $k(x) = \exp\left[\mp\sqrt{2(B+\epsilon)}x\right]$, that is, let X correspond to the semilinear elliptic operator $\mathcal{A}$, where

$$\mathcal{A}(u) := \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - Bu - \exp\left[\mp\sqrt{2(B+\epsilon)}x\right] u^2.$$

By Theorem 1.6, X suffers weak extinction:

$$\lim_{t\to 0} e^{(B-\delta)t}\|X_t\| = 0.$$

Also, clearly, $\lambda_2 = -B$, yielding that X also exhibits local extinction.

Now we are going to show that, despite the above, the process X survives with positive probability, that is

$$\mathbb{P}_\mu(\|X_t\| > 0,\ \forall\, t > 0) > 0,$$

for any $\mathbf{0} \neq \mu \in M(\mathbb{R}^d)$. In order to do this, we will use the definition and basic properties of h-transforms and weighted superprocesses. These can be found in Section 2 of [9].

The function $h(x) := e^{\pm\sqrt{2(B-\epsilon)}x}$ transforms the operator $\mathcal{A}$ into $\mathcal{A}^h$, where

$$\mathcal{A}^h(u) := \frac{1}{h}\mathcal{A}(hu) = \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \pm \sqrt{2(B+\epsilon)}\frac{\mathrm{d}u}{\mathrm{d}x} + \epsilon u - u^2.$$

(Note that $h''/2 - (B+\epsilon)h = 0$). The superprocess X^h corresponding to $\mathcal{A}^h$ is in fact the same as the original process X, weighted by the function h, and consequently, survival (with positive probability) is invariant under h-transforms. But X^h has a conservative motion component and constant branching mechanism, which is supercritical, and therefore X^h survives with positive probability; the same is then true for X. $\diamond$

5.3 The super-Brownian motion case

In this subsection we focus on the special case when the underlying motion process is a Brownian motion, that is, when $L = \Delta/2$; in the remainder of this section we will always assume that this is the case. In this case $\beta \in \mathbf{K}(\xi)$ if and only if

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^d}\int_{|y-x|<r} u(x-y)|\beta(y)|\,\mathrm{d}y = 0,$$

where u is the function defined in (2.4). When $d \geq 3$, $\mathbf{K}_\infty(\xi)$ coincides with the class $\mathbf{K}_d^\infty$ defined in [30]. We recall the definition of the class $\mathbf{K}_d^\infty$ defined in [13, 14] in the case $d \leq 2$.

Definition 5.1 (The classes $\mathbf{K}_1^\infty(\xi)$ and $\mathbf{K}_2^\infty(\xi)$) Let $L = \Delta/2$.

(1) If $d = 1$, a function $q \in \mathbf{K}(\xi)$ *is said to be in the class* $\mathbf{K}_1^\infty(\xi)$ if

$$\int_{|y|\geq 1} |yq(y)|\mathrm{d}y < \infty.$$

(2) If $d = 2$, a function $q \in \mathbf{K}(\xi)$ is *said to be in the class* $\mathbf{K}_2^\infty(\xi)$ if

$$\int_{|y|\geq 1} \ln(|y|)|q(y)|\mathrm{d}y < \infty.$$

5.3.1 The $d \geq 3$ case

We first recall the following definition from [25].

Definition 5.2 (Criticality in terms of λ_∞) Let $L = \Delta/2$ and $\beta \in \mathbf{K}(\xi)$. Then β is said to be

(a) *supercritical* iff $\lambda_\infty(\beta) > 0$,

(b) *subcritical* iff $\lambda_\infty(\beta) = 0$, and $\lambda_\infty((1+\epsilon)\beta) = 0$ for some $\epsilon > 0$.

(c) *critical* iff $\lambda_\infty(\beta) = 0$ and $\lambda_\infty((1+\epsilon)\beta) > 0$ for all $\epsilon > 0$.

Note: The reader should not confuse the above properties of the function β with the (local) criticality (or sub- or supercriticality) of the branching, which simply refer to the sign of β (in certain regions).

The following result relates the above definition to the solutions of

$$(L + \beta)u = 0, \tag{5.3}$$

and is due to [30].

Lemma 5.1 *Let $L = \Delta/2$, $\beta \in \mathbf{K}_\infty(\xi)$ and $d \geq 3$. Then the following conditions are equivalent:*

(a) β is subcritical.

(b) $g_\beta(x) \equiv \Pi_x e_\beta(\infty)$ is bounded in $\mathbb{R}^d$.

(c) There exists a solution u to (5.3) with $\inf_{x \in \mathbb{R}^d} u(x) > 0$.

(d) There exists a solution u to (5.3) with $0 < \inf_{x \in \mathbb{R}^d} u(x) \leq \sup_{x \in \mathbb{R}^d} u(x) < \infty$.

Moreover, if β is subcritical, then (5.3) has a unique (up to constant multiples) positive bounded solution and the solution must be of the form $c g_\beta(x)$ for some $c > 0$.

However, if $\beta - \lambda_\infty$ is critical, there is no positive solution bounded away from 0. Pinchover [21] proved the following result (see [21, Lemma 2.7]).

Lemma 5.2 *Let $L = \Delta/2$, $\beta \in \mathbf{K}_\infty(\xi)$ and $d \geq 3$. If β is critical, then there is an $h > 0$ satisfying (5.3) on $\mathbb{R}^d$ and such that*

$$h \sim c_d |x|^{2-d}, \quad \textit{as } |x| \to \infty, \tag{5.4}$$

where c_d is a positive constant depending only on d.

It is easy to check that, for any $p > d/2$, $\beta \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ implies that $\beta \in \mathbf{K}_\infty(\xi)$. In this special case, the following result shows that h can be obtained as large time asymptotic limit of Schrödinger semigroup (see [25, Theorem 3.1]

Lemma 5.3 *Let $L = \Delta/2$, $\beta \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $d \geq 3$. If β is critical, then*

$$\lim_{t\to\infty} f(t)^{-1} \sup_{x\in\mathbb{R}^d} \Pi_x[e_\beta(t)] = C, \tag{5.5}$$

and

$$\lim_{t\to\infty} f(t)^{-1} \Pi_x[e_\beta(t)] = h(x), \quad \forall x \in \mathbb{R}^d, \tag{5.6}$$

where C is a positive constant, $h > 0$ is bounded and solves (5.3) (general theory implies, in the critical case, the existence of such a solution) and

$$f(t) = \begin{cases} t, & d \geq 5, \\ t/(\ln t), & d = 4, \\ t^{1/2} & d = 3. \end{cases} \tag{5.7}$$

Lemma 5.4 *Let $L = \Delta/2$ and $d \geq 3$. If $\lambda_\infty(\beta) > 0$ and $\beta - \lambda_\infty \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, then conditions* (1.11) *and* (1.13) *are satisfied.*

Proof. Note that

$$g_\beta(t) = \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t) = e^{\lambda_\infty t} \sup_{x\in\mathbb{R}^d} \Pi_x e_{\beta-\lambda_\infty}(t).$$

By Lemma 5.3 we have

$$g_\beta(t) \sim C e^{\lambda_\infty t} f(t), \quad \text{as } t \to \infty$$

with $f(t)$ defined by (5.7), and

$$\lim_{t\to\infty} g_\beta^{-1}(t) \Pi_x e_\beta(t) = \frac{1}{C} \lim_{t\to\infty} f^{-1}(t) \Pi_x e_{\beta-\lambda_\infty}(t) > 0,$$

which means that conditions (1.11) and (1.13) are satisfied. □

5.3.2 The $d \leq 2$ case

The following Lemma is due to [13, 14].

Lemma 5.5 *Let $d \leq 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty(\xi)$. The following conditions are equivalent.*

(a) β is critical.

(b) There exists a positive bounded solution to (5.3).

Moreover, if β is critical, the positive bounded solution h to (5.3) is unique (up to constant multiples), and h possesses the following representation:

$$h(x) = \begin{cases} h(0)\lim_{r\downarrow 0}\Pi_x e_\beta(T_{B(0,r)}), & d=2 \\ h(0)\Pi_x e_\beta(T_0), & d=1. \end{cases}$$

where for every open set B, $T_B = \inf\{t>0; \xi_t \in B\}$ denotes the first hitting time of B, and $T_0 = T_{\{0\}}$ denotes the first hitting time of ξ at the point 0. h is also bounded below from 0.

It follows from the lemma above that, in the case $d \le 2$, if $\lambda_\infty(\beta) > 0$, $\beta - \lambda_\infty(\beta) \in \mathbf{K}_d^\infty$ and $\beta - \lambda_\infty(\beta)$ is critical, then the assumption (1.18) of Theorem 1.8 is satisfied.

Remark 5.6 *Let $d \le 2$ and $L = \Delta/2$. If $\beta \sim |x|^{-\rho}$ ($\rho > 4$) as $|x| \to \infty$ (obviously $\beta \in \mathbf{K}_d^\infty$) and β is subcritical, Murata proved that there exists a positive solution h to (5.3) such that*

$$h(x) = \begin{cases} (2\pi)^{-1}\log\frac{|x|}{2} + \mathcal{O}(1), & \text{for } d=2, \\ (2\pi^{1/4})^{-1}|x| + \mathcal{O}(1), & \text{for } d=1, \end{cases}$$

as $|x| \to \infty$. See [20, Theorem 4.1]. ◇

Thus if $d \le 2$, $L = \Delta/2$, $\beta - \lambda \in \mathbf{K}_d^\infty$ and $\beta - \lambda$ is subcritical, then there is no positive bounded solution to $(L + \beta - \lambda)h = 0$. In order to deal with the subcritical case, we need to develop some results on Schrödinger semigroups. We believe, that these results are also of independent interest.

Lemma 5.7 *Let $d \le 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If $\lambda_\infty(\beta) = 0$, then*

$$\sup_{t\ge 0}\sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t) < \infty. \tag{5.8}$$

Proof. Since $\lambda_\infty(\beta) = 0$, β is either critical or subcritical. For the subcritical case we will prove a stronger result later, see Lemma 5.7. Now we suppose that β is critical. Then

Lemma 5.5 asserts that there exists a bounded solution ψ to (5.3) such that $\psi > 0$ and $\sup_{x\in\mathbb{R}^d} \psi^{-1}(x) < \infty$. Then we have

$$\begin{aligned}\Pi_x e_\beta(t) = & \ \Pi_x(e_\beta(t)(\psi^{-1}\psi)(\xi_t)) \\ \leq & \ (\sup_{x\in\mathbb{R}^d} \psi^{-1}(x))\, \Pi_x(e_\beta(t)\psi(\xi_t)) \\ = & \ (\sup_{x\in\mathbb{R}^d} \psi^{-1}(x))\, \psi(x) \\ \leq & \ \sup_{x\in\mathbb{R}^d} \psi(x) / \inf_{x\in\mathbb{R}^d} \psi(x) < \infty.\end{aligned}$$

This proves (5.8). $\square$

Remark 5.8 *Murata (see [20, Corollary 1.6]) proved the above result for $d = 2$ under the condition that $\beta \sim |x|^{-\rho}$ $(\rho > 4)$ as $|x| \to \infty$, which implies that $\beta \in \mathbf{K}_2^\infty$. Our proof above goes along the line given in the proof of [20, Corollary 1.6(ii)].* $\diamond$

If β is subcritical, we have the following stronger result.

Lemma 5.9 *Let $d \leq 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If β is subcritical, then*

$$\sup_{x\in\mathbb{R}^d} \Pi_x \sup_{0\leq t\leq\infty} e_\beta(t) < \infty. \tag{5.9}$$

Proof. We first prove the result for dimension $d = 2$. For $r > 0$ we denote the open ball of radius r with center at the origin and its open exterior by

$$B_r = \{x \in \mathbb{R}^d, \quad |x| < r\}; \quad B_r^* = \{x \in \mathbb{R}^d, \quad |x| > r\}.$$

According to [14, Proposition 2.2], there exists an $r_0 > 0$ such that for all $r \geq r_0$ and $x \in B_r^*$,

$$\Pi_x e_{\beta^+}(\tau_{B_r^*}) \leq 2, \quad e^{-1/2} \leq \Pi_x e_\beta(\tau_{B_r^*}) \leq 2. \tag{5.10}$$

Choose r_0 large enough such that $\mathrm{Supp}\mu \subset B_{r_0}$. We fix two real numbers r and R with $R > r \geq r_0$. Since β is subcritical, by [13, Theorem 2.1],

$$\Pi_x e_\beta(\tau_{B_R}) < \infty, \quad \forall x \in B_R.$$

We define

$$S = \tau_{B_R} + \tau_{B_r^*} \circ \theta_{\tau_{B_R}}.$$

Put

$$S_0 = 0; \quad S_n = S_{n-1} + S \circ \theta_{S_{n-1}}, \quad n \geq 1.$$

In particular, $S_1 = S$. For any $f \in C(\partial B_r)$, we define

$$(A_S f)(x) = \Pi_x(e_\beta(S) f(\xi_S)), \quad x \in \partial B_r.$$

Note that

$$A_S^n f(x) = \Pi_x \left[e_\beta(S_n) f(S_n) \right], \quad x \in \partial B_r.$$

The spectral radius of A_S is defined by

$$\widetilde{\lambda}(\beta) := \lim_{n\to\infty} \|A_S^n\|^{1/n}.$$

It follows from [14, Theorem 2.4] that $\widetilde{\lambda}(\beta) < 1$. Thus there exists $\delta > 0$ such that $\widetilde{\lambda}(\beta) + \delta < 1$, and sufficiently large n such that, $\|A_S^n\| \le (\widetilde{\lambda}(q) + \delta)^n$. Therefore we have

$$\sum_{n=0}^{\infty} \sup_{x\in\mathbb{R}^d} |A_S^n 1(x)| = \sum_{n=0}^{\infty} \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(S_n) < \infty. \tag{5.11}$$

By the strong Markov property applied at τ_{B_R}, and by (5.10), we have

$$\begin{aligned}
\Pi_x \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t = {} & \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x \left[\Pi_{\xi_{\tau_{B_R}}} \int_0^{\tau_{B_r^*}} e_\beta(t)\beta^+(t)\,\mathrm{d}t \right] \\
\le {} & \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x \left[\Pi_{\xi_{\tau_{B_R}}} \int_0^{\tau_{B_r^*}} e_{\beta^+}(t)\beta^+(t)\,\mathrm{d}t \right] \\
= {} & \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x \left[\Pi_{\xi_{\tau_{B_R}}} e_{\beta^+}(\tau_{B_r^*}) \right] - 1 \\
\le {} & \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + 1.
\end{aligned}$$

Let ξ^{B_R} denote the Brownian motion killed upon exiting B_R. Since β is subcritical, $(\xi^{B_R}, \mathbf{1}_{B_R}\beta)$ is gaugeable, where $\mathbf{1}_{B_R}$ is the indicator operator on B_R. It follows from [2, Theorem 2.8] that

$$\sup_{x\in B_R} \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t < \infty.$$

Thus

$$C := \sup_{x\in\partial B_r} \Pi_x \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t < \infty. \tag{5.12}$$

By the strong Markov property, applied at S_n, and by (5.11), and (5.12), we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \Pi_x \int_0^\infty e_\beta(t)\beta^+(t)\,\mathrm{d}t \le {} & \textstyle\sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x \left[\int_{S_n}^{S_{n+1}} e_\beta(t)\beta^+(t)\,\mathrm{d}t \right] \\
= {} & \textstyle\sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x \left[e_\beta(S_n) \Pi_{\xi_{S_n}} \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t \right] \\
\le {} & C \textstyle\sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(S_n) < \infty.
\end{aligned} \tag{5.13}$$

Observe that

$$e_\beta(t) = 1 + \int_0^t e_\beta(s)\beta(s)\,\mathrm{d}s \le 1 + \int_0^t e_\beta(s)\beta^+(s)\,\mathrm{d}s,$$

and so

$$\sup_{0\le t\le\infty} e_\beta(t) \le 1 + \int_0^\infty e_\beta(s)\beta^+(s)\,\mathrm{d}s.$$

Using (5.13) we get (5.9) and we finish the proof for dimension $d = 2$.

Now suppose $d = 1$. Define

$$u(a, b) = \Pi_x e_\beta(T_b), \quad a, b \in \mathbb{R}^1,$$

where T_b is the first hitting time of ξ at the point b. By Theorem 4.8 in [13], $u(a, b)u(b, a) < 1$ for any $a, b \in \mathbb{R}^1$. For any $x \in \mathbb{R}^1$, define

$$S_x = T_{x+1} + T_x \circ \theta_{T_{x+1}}.$$

Then

$$\Pi_x e_\beta(S_x) = u(x, x+1)u(x+1, x) < 1.$$

Repeating the above proof for $d = 2$ with S replaced by S_x we can similarly obtain (5.9) for $d = 1$. We omit the details. □

Lemma 5.10 *Let $d \le 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If β is subcritical, then*

$$\lim_{t\to\infty} \Pi_x e_\beta(t) = \Pi_x e_\beta(\infty) \equiv 0 \quad \text{in } \mathbb{R}^d. \tag{5.14}$$

Proof. By (5.9) and by dominated convergence, it suffices to show

$$\Pi_x e_\beta(\infty) = 0, \quad \forall x \in \mathbb{R}^d. \tag{5.15}$$

We continue to use the notations in the proof of Lemma 5.9. We first prove (5.15) for dimension $d = 2$. Using the strong Markov property of ξ, applied at τ_{B_R}, and Fatou's lemma, we get

$$\begin{aligned}
\Pi_0 e_\beta(\infty) = \;& \Pi_0\left[e_\beta(\xi_{\tau_{B_r}})\Pi_{\xi_{\tau_{B_r}}} e_\beta(\infty)\right] \\
\le \;& \Pi_0\left[e_\beta(\tau_{B_r})\lim_{n\to\infty}|(A_S^n)1(\xi_{\tau_{B_r}})|\right] \\
\le \;& [\Pi_0 e_\beta(\tau_{B_r})]\lim_{n\to\infty}\|A_S^n\| \\
\le \;& [\Pi_0 e_\beta(\tau_{B_r})]\lim_{n\to\infty}(\widetilde{\lambda}(\beta)+\delta)^n = 0.
\end{aligned}$$

Thus by Lemma 2.3, $\Pi_x e_\beta(\infty) \equiv 0$ in $\mathbb{R}^2$.

Now we suppose $d = 1$. For any $x \in \mathbb{R}$, let S_x be defined as in proof of Lemma 5.9. By the strong Markov property of ξ applied at S_x, we have, for any $x \in \mathbb{R}^1$,

$$\Pi_x e_\beta(\infty) = \Pi_x e_\beta(S_x) \Pi_x e_\beta(\infty).$$

Since $\Pi_x e_\beta(S_x) = u(x, x+1)u(x+1, x) < 1$, the above equality yields $\Pi_x e_\beta(\infty) = 0$ for every $x \in \mathbb{R}$. $\square$

Remark 5.11 *It follows from the two results above that, if $d \leq 2$, $L = \Delta/2$, $\lambda_\infty(\beta) > 0$, $\beta - \lambda_\infty(\beta) \in \mathbf{K}_d^\infty$ and $\beta - \lambda_\infty(\beta)$ is subcritical, then the assumptions of Theorem 1.7(2) are satisfied.*

5.4 Compactly supported mass annihilation

We conclude this section of examples, as well as the whole article, with two simple examples which satisfy the assumptions of Theorem 1.7(2). In both cases we consider compactly supported mass annihilation terms.

We start with a two-dimensional example.

Example 5.4 (d=2; constant annihilation in a compact) Let ξ be planar Brownian motion, and $\beta(x) := -\alpha \mathbf{1}_K(x)$ with $\alpha > 0$ being a constant and $K \subset \mathbb{R}^2$ a compact with non-empty interior.

Proposition 5.12 *In this case weak extinction holds.*

Proof: It is well known that β is subcritical (see, e.g., [20, Theorem 1.4]). By [1, Corollary 2], as $t \to \infty$,

$$\Pi_x \left[\exp\left(\int_0^t \beta(\xi_s)\, \mathrm{d}s \right) \right] \sim c(\log t)^{-1},$$

where c is a positive constant determined by x, K and α. Therefore, for any $x \in \mathbb{R}^2$, $\lambda_\infty(\beta) \geq \lim_{t\to\infty} \frac{1}{t} \log \Pi_x e_\beta(t) = 0$. It is obvious that $\lambda_\infty(\beta) \leq 0$. Then $\lambda_\infty = 0$ and $g_{\beta-\lambda_\infty}(x) \equiv 0$. It is obvious that (1.15) holds since $\beta \leq 0$. Using again that $\beta \leq 0$, we are done by part (2) of Theorem 1.7. $\square$

Finally, we discuss an example in one-dimension.

Example 5.5 (d=1; compactly supported mass annihilation) Let ξ be a Brownian motion in $\mathbb{R}$, and $\beta \leq 0$ a continuous function on $\mathbb{R}$ with compact support.

Proposition 5.13 *In this case weak extinction holds.*

Proof: It is well known that β is subcritical (see [24]). By [29],

$$\lim_{t\to\infty} t^{-1/2} \int_0^t \beta(\xi_s)\,\mathrm{d}s = \eta \int_{-\infty}^{\infty} \beta(x)\mathrm{d}x, \tag{5.16}$$

in distribution, where η is a random variable with $\eta \neq 0$ a.s. This implies that

$$\lim_{t\to\infty} \Pi_x \exp\left(t^{-1/2} \int_0^t \beta(\xi_s)\,\mathrm{d}s\right) = \Pi_x \exp(a\eta),$$

where $a = \int_{-\infty}^{\infty} \beta(x)\,dx$. Using Jensen's inequality, we get

$$\liminf_{t\to\infty} \left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right]^{t^{-1/2}} \geq \Pi_x \exp(a\eta),$$

which implies that

$$\liminf_{t\to\infty} t^{-1/2} \log\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] \geq \log \Pi_x \exp(a\eta).$$

Thus

$$\liminf_{t\to\infty} t^{-1} \log\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] \geq 0.$$

The assumption $\beta \leq 0$ implies that

$$\limsup_{t\to\infty} t^{-1} \log\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] \leq 0.$$

Hence

$$\lambda_\infty = \lim_{t\to\infty} t^{-1} \log\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] = 0.$$

By (5.16), we have

$$g_{\beta-\lambda_\infty}(x) = \Pi_x \exp\left(\int_0^{\infty} \beta(\xi_s)\,\mathrm{d}s\right) \equiv 0.$$

As before, (1.15) holds since $\beta \leq 0$. The proof is then finished exactly like in the proof of Example 5.4. $\square$

6 Appendix

Proof of Theorem 1.2 Suppose $D_n, n \ge 1$, is a sequence of smooth bounded domains such that $D_n \uparrow \mathbb{R}^d$. According to Dynkin [7], for each n, the $(L|_{D_n} - \beta^-, \beta^+ \wedge n, k)$-superdiffusion $(X_t^n, t \ge 0)$ exists, where $L|_{D_n}$ is the generator of the process ξ killed upon leaving D_n, and β^+ and β^- are the positive and negative parts of β, respectively. Also note that $(X_t^n, t \ge 0)$ can be regarded as a $(L|_{D_n}, \beta \wedge n, k)$-superdiffusion.

Let f be a positive bounded measurable function on $\mathbb{R}^d$. According to Dynkin [7], for each n, there exist unique bounded solution u_n to the following integral equation:

$$u_n(t,x) + \Pi_x \int_0^{t\wedge\tau_n} [-(\beta(\xi_s) \wedge n) u_n(t-s, \xi_s) + k(\xi_s) u^2(t-s, \xi_s)] \mathrm{d}s = \Pi_x[f(\xi_t), t < \tau_n],$$

where τ_n is the first exit time of the diffusion ξ from D_n. We rewrite the above equation in the following form (according to a result similar to our Lemma 3.1):

$$u_n(t,x) + \Pi_x \int_0^{t\wedge\tau_n} e_{\beta^+\wedge n}(s)[\beta^-(\xi_s) u_n(\xi_s, t-s) + k(\xi_s) u^2(\xi_s, t-s)] \mathrm{d}s = \Pi_x[e_{\beta^+\wedge n}(t) f(\xi_t), t < \tau_n]. \tag{6.1}$$

By the (weak) parabolic maximum principle (see [19, p. 128] for example), u_n is increasing. Let $u_n(t,x) \uparrow u(t,x)$ as $n \uparrow \infty$. Letting $n \to \infty$ in the above integral equation, we get

$$u(t,x) + \Pi_x \int_0^t e_{\beta^+}(s)[\beta^-(\xi_s) u(t-s, \xi_s) + k(\xi_s) u^2(t-s, \xi_s)] \mathrm{d}s = \Pi_x[e_{\beta^+}(t) f(\xi_t)] \tag{6.2}$$

The assumption that β is in Kato class implies that $u(t,x) \le \Pi_x[e_{\beta^+}(t) f(\xi_t)] \le e^{c_1 + c_2 t}$ for some positive constants.

To see the minimality of u, let v be an arbitrary nonnegative measurable solution to (6.2). By the (weak) parabolic maximum principle, $v|_{D_n} \ge u_n$ for all $n \ge 1$, and thus $v \ge u$ on $\mathbb{R}^d$.

Equation (6.2) can be rewritten as

$$u(t,x) + \Pi_x \int_0^t [-\beta(\xi_s) u(t-s, \xi_s) + k(\xi_s) u^2(t-s, \xi_s)] \mathrm{d}s = \Pi_x[f(\xi_t)]. \tag{6.3}$$

Then following the arguments in Appendix A of Engländer and Pinsky [9], we can get the existence of our superdiffusion. □

Remark 6.1 If $k \in \mathbf{K}(\xi)$ as well, then using Gronwall's lemma, u is the *unique* solution (bounded on any finite interval) of the integral equation (6.3).

Before turning to the proof Theorem 1.4, we remark that [18, Appendix A] explains some important concepts (e.g. Ray cone, Ray topology) we will be working with, and that [18, Chap. 5] discusses regularity properties of superdiffusions, using similar methods, albeit under different assumptions on the nonlinear operator.

For the proof we first need a lemma. The function f is called[1] *α-supermedian* relative to P_t^0 for $\alpha > 0$, if $e^{-\alpha t} P_t^0 f \le f$ for $t \ge 0$.

Lemma 6.2 *Assume that $\beta \in \mathbf{K}(\xi)$ satisfies $\beta \le B$ for some constant $B > 0$, and f is α-supermedian relative to P_t^0 for some $\alpha > 0$. Then for every $\mu \in M(\mathbb{R}^d)$,*

(i) $M_t := e^{-(B+\alpha)t}\langle f, X_t\rangle$ is a $\mathbb{P}_\mu$-supermartingale.

(ii) $\mathbb{P}_\mu\left(\sup_{0\le r\le t, r\in\mathbb{Q}}\langle 1, X_t\rangle < \infty \text{ for all } t > 0\right) = 1$.

Proof. (i) It is easy to see that it suffices to check

$$\mathbb{E}_\nu(M_t) \le M_0 = \langle f, \nu\rangle,\ t > 0,\ \forall \nu \in M(\mathbb{R}^d). \tag{6.4}$$

This is because for $0 \le s < t$, by the Markov property at time s,

$$\mathbb{E}_\mu\left(e^{-Bt}\langle f, X_t\rangle \mid \mathcal{F}_s\right) = \mathbb{E}_{X_s} M_{t-s} e^{-(B+\alpha)s} \le \langle f, X_s\rangle e^{-(B+\alpha)s} = M_s,$$

where in the last inequality above we used (6.4) with $\nu = X_s$. Using the assumption that f is α-supermedian, we obtain

$$\mathbb{E}_{\delta_x} M_t = e^{-(B+\alpha)t}(P_t^\beta f)(x) \le e^{-\alpha t} P_t^0 f(x) \le f(x).$$

Therefore (6.4) holds.

(ii) By the proof of Theorem 1.6, there are $a, \gamma > 0$ and a sufficiently small $T > 0$ such that $M_r := e^{\gamma t}\langle 1, X_r\rangle$ satisfies

$$\mathbb{P}_\mu[M_r|\mathcal{F}_s] \ge a M_s, \quad 0 \le s \le r \le T \text{ with } r, s \in \mathbb{Q}.$$

Then by Doob's inequality (Lemma 3.3 in discrete time),

$$\mathbb{P}_\mu\left(\sup_{0\le r\le T,\, r\in\mathbb{Q}} \langle 1, X_r\rangle > K\right) \le (aK)^{-1}\mathbb{P}_\mu M_t \le (aK)^{-1} e^{(\gamma+B)T}.$$

Letting $K \uparrow 0$, we see that for any fixed $t > 0$, $\mathbb{P}_\mu(\sup_{0\le r\le T,\, r\in\mathbb{Q}}\langle 1, X_r\rangle = \infty) = 0$. Since we can split $[0, \infty)$ to intervals of length T, the result of (ii) holds. $\square$

Proof of Theorem 1.4 Let $(\overline{\mathbb{R}}^d, \overline{\mathcal{B}(\mathbb{R}^d)})$ be the Ray-Knight compactification of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ associated with the semigroup $\{P_t^0 : t \ge 0\}$ and a suitably chosen countable

[1] In [18] a slightly different terminology is followed.

Ray cone (see the last paragraph on [12, p. 342]), and let $M_r(\overline{\mathbb{R}}^d)$ be the space of finite measures on $\overline{\mathbb{R}}^d$ with the weak Ray topology. Suppose W is the space of right continuous paths from $[0,\infty)$ to $M_r(\mathbb{R}^d)$ with left limits in $M_r(\overline{\mathbb{R}}^d)$, where $M_r(\mathbb{R}^d)$ carries the relative topology inherited from $M_r(\overline{\mathbb{R}}^d)$. We write $\tilde{X} = (\tilde{X}_t, t \geq 0)$ for the coordinate process on W and put $\mathcal{G} = \sigma\{\tilde{X}_t; t \geq 0)$. Using the above lemma, the argument in the proof of [12, Theorem 2.11] is applicable to our setup, so for any given $\mu \in M(\mathbb{R}^d)$ there exists a unique probability measure $\mathbf{P}_\mu$ on $(W, \mathcal{G})$ such that $\mathbf{P}_\mu(\tilde{X}_0 = \mu) = 1$ and $(\tilde{X}_t, t \geq 0)$ under $\mathbf{P}_\mu$ has the same law as the superprocess X under $\mathbb{P}_\mu$.

Let $M_0(\mathbb{R}^d)$ be the space of finite measures on $\mathbb{R}^d$ with the weak topology induced by the mappings $\langle f, \tilde{X}_t\rangle$ as f runs through the bounded continuous functions on $\mathbb{R}^d$. (The Borel σ-algebras on $M_r(\mathbb{R}^d)$ and $M_0(\mathbb{R}^d)$ both coincide with $\mathcal{M}$.) Since the diffusion process ξ is continuous, using the arguments of [12, Section 3], we have that if f is a bounded continuous function on $\mathbb{R}^d$, then $\langle f, \tilde{X}_\cdot\rangle$ is right continuous on $[0,\infty)$ almost surely; and if $f(\xi_\cdot)$ has left limits on $[0,\infty)$ almost surely, then so does $\langle f, \tilde{X}_\cdot\rangle$. That is to say $\tilde{X}$ is a *càdlàg* process on the state space $M_0(\mathbb{R}^d)$. $\square$

Acknowledgement The first author owes thanks to Zenghu Li for valuable discussions about path regularity questions.

References

[1] Chan T. (1994). Occupation times of compact sets by planar Brownian motion, Ann. Inst. H. Poincaré Probab. Statist., **30**: 317–329.

[2] Chen, Z.-Q. (2002). Gaugeability and conditional Gaugeability, Trans. Amer. Math. Soc., **354**: 4639–4679.

[3] Chen, Z.-Q. (2011). Uniform integrability of exponential martingales and spectral bounds of non-local Feynman-Kac semigroups. Preprint.

[4] Chen, Z.-Q. and Song, R. (2002). General gauge and conditional gauge theorems, Ann. Probab., **30**: 1313–1339.

[5] Chung, K. L. and Zhao, Z. (1995). From Brownian Motion to Schrödinger's equation, Springer-Verlag, Berlin, Heidelberg.

[6] Durrett, R. (1996). Probability Theory and Examples, Second Edition, Duxbury Press.

[7] Dynkin, E. B. (1993) Superprocesses and partial differential equations, Ann. Probab., **21**: 1185–1262.

[8] Engländer, J. and Kyprianou, A. E. (2004). Local extinction versus local exponential growth for spatial branching processes, Ann. Probab., **32**: 78–99.

[9] Engländer, J. and Pinsky, R. G. (1999). On the construction and support properties of measure-valued diffusions on $D \subset \mathbb{R}^d$ with spatially dependent branching, Ann. Probab., **27**: 684–730.

[10] Engländer, J. and Turaev, D. (2002). A scaling limit theorem for a class of superdiffusions, Ann. Probab., **30**: 286–722.

[11] Engländer, J. and Winter, A. (2006). Law of large numbers for a class of superdiffusions, Ann. I. H. Poincaré-PR, **42**: 171–185.

[12] Fitzsimmons, P. J. (1988). Construction and regularity of measure-valued Markov branching processes, Israel J. Math., **64**: 337–361.

[13] Gesztesy, F. and Zhao, Z. (1991). On critical and subcritical Sturm-Liouville operators, J. Funct. Anal., **98**: 311–345.

[14] Gesztesy, F. and Zhao, Z. (1995). On positive solutions of critical Schrödinger operator in two dimension, J. Funct. Anal., **127**: 235–256.

[15] Hueber, H. and Sieveking, M. (1982). Uniform bounds for quotients of Green functions on $C^{1,1}$-domains, Ann. Inst. Fourier (Grenoble), **32**: 105-117.

[16] Kim, P. and Song, R. (2006). Two-sided estimates on the density of Brownian motion with singular drift, Illinois J. Math., **50** 635–688

[17] Kim, P. and Song, R. (2008). On dual processes of non-symmetric diffusions with measure-valued drifts. Stoch. Proc. Appl., **118**: 790–817

[18] Li, Z. (2011) Measure-valued branching Markov processes. Probability and its Applications (New York). Springer, Heidelberg.

[19] Lieberman, G. M. (1996) Second order parabolic differential equations, World Scientific Publishing Co. Inc., River Edge, NJ.

[20] Murata, M. (1984). Positive solutions and large time behaviour of Schrödinger semigroup, Simon's problem, J. Funct. Anal.,**56**: 300–310.

[21] Pinchover, Y. (1995) On the localization of binding for Schrödinger operators and its extension to elliptic operators, J. Analyse Math., **66**: 57–83.

[22] Pinsky, R. G. (1995) Positive Harmonic Functions and Diffusion, Cambridge Univ. Press.

[23] Pinsky, R. G. (1996) Transience, recurrence and local extinction properties of the support for supercritical finite measure-valued diffusions. Ann. Probab., **24**: 237–267.

[24] Reed, M. and Simon, B. (1978) Methods of Modern Mathematical Physics, IV, Analysis of Operators, Academic Press, New York

[25] Simon, B. (1981). Large time behavior of the L^p norm of Schrödinger semigroups, J. Funct. Anal., **40**: 66–83.

[26] Song, R. and Vondracek, Z. (2004). Harnack inequality for some classes of Markov processes. Math. Z., **246**: 177–202.

[27] Stroock, D. W. (2011). Probability Theory: An Analytic View. Second Edition. Cambridge University Press.

[28] Stroock, D. W. and Varadhan, S. R. S. (1997). Multidimensional diffusion processes, Springer, Berlin.

[29] Yamada, T. (1986). On some limit theorems for occupation times of one-dimensional Brownian motion and its continuous additive functionals locally of zero energy J. Math. Kyoto Univ. **26**: 309–322.

[30] Zhao, Z. (1992) Subcriticality and gaugeability of the Schrodinger operator, Trans. Amer. Math. Soc., **334**: 75–96.

János Engländer: Department of Mathematics, University of Colorado, Boulder, CO 80309-0395, U.S.A. Email: `janos.englander@colorado.edu`

Yan-Xia Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. Email: `yxren@math.pku.edu.cn`

Renming Song: Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A. Email: `rsong@math.uiuc.edu`